# THE TWISTOR LINE

ABSTRACT. Over $\mathbf{C}$, the $n$th singular cohomology group of a variety has a canonical mixed Hodge structure. If we restrict to smooth projective varieties, we get a pure Hodge structure.

It was observed by Simpson that the category of pure Hodge structures can be formulated geometrically as a category of vector bundles on a curve, and so can the association of Hodge structure to variety. I'll talk about how this is done using the twistor line.

## 1. HODGE THEORY REVIEW

Hodge theory in general can tell us about varieties over $\mathbf{C}$, but in this talk I'll stick to smooth projective varieties since that makes everything a lot easier to state.

When I have a smooth projective variety $X$ over $\mathbf{C}$, I can take the associated complex manifold $X^{\mathrm{an}}$ to $X(\mathbf{C})$. This is realized as a complex submanifold $X^{\mathrm{an}} \subseteq \mathbf{P}^n(\mathbf{C})$, which implies it is compact.

It is also going to be Kähler, which means there is a Hermitian metric $h$ on $TX^{\mathrm{an}}$ (that is, we put a smoothly varying positive definite Hermitian form on each fiber). Kähler means the associated 2-form $\omega = \Im(h(v, w))$ is closed. You can think of a complex manifold being Kähler as giving the data of compatible Riemannian, complex, and symplectic structures.

We can also see this from the fact that $X^{\mathrm{an}}$ sits inside projective space as a complex submanifold: we can use the Fubini-Study form to make $\mathbf{P}^n(\mathbf{C})$ a Kähler manifold, so by restricting the Káhler form we obtain one on $X^{\mathrm{an}}$.

In this restricted setting, we have the following important theorem.

> THEOREM 1.1. Let $X$ be a compact Kähler manifold. Then
> $$\mathrm{H}^n_{\mathrm{sing}}(X, \mathbf{C}) \simeq \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$
> where $\mathrm{H}^{p,q} \simeq \mathrm{H}^q(X, \Omega^p)$. We call the direct sum *Dorbealt cohomology*.
>
> Moreover, if $X$ arises from a smooth projective variety, we can compute $\mathrm{H}^q(X, \Omega^p)$ algebraically.

This is interesting, since the singular cohomology is topological in nature and the Dorbealt cohomology is algebraic in nature. The numbers $h^{p,q} := \dim_{\mathbf{C}} \mathrm{H}^q(X, \Omega^p)$ are called the

Hodge numbers. We note that these exhibit two general symmetries: $h^{p,q} = h^{q,p}$, and $h^{N-p,N-q} = h^{p,q}$ for $\dim_{\mathbf{C}} \mathrm{X} = N$.

This decomposition allows us to write down what is called an *integral Hodge structure*.

> DEFINITION 1.2. A pure integral Hodge structure of weight $n$ is a pair $(\mathrm{H}_{\mathbf{Z}}, \mathrm{H}^{p,q})$ consisting of a free finitely generated abelian group $\mathrm{H}_{\mathbf{Z}}$ and a decomposition
>
> $$\mathrm{H}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$
>
> where $\mathrm{H}^{p,q} = \overline{\mathrm{H}^{q,p}}$.
>
> Similarly, one defines rational and real Hodge structures.

There is a canonical way to attach such structures to compact Kähler manifolds: you take $\mathrm{H}^n(\mathrm{X}, \mathbf{Z})/\mathrm{tors}$, and then when we tensor with $\mathbf{C}$ you get the Hodge structure from the Hodge decomposition theorem.

Note that this association is functorial, in that given a holomorphic map $f$ we get a map of the associated Hodge structures: this is a group homomorphism $\mathrm{V}_{\mathbf{Z}} \to \mathrm{W}_{\mathbf{Z}}$, such that $\mathrm{V}^{p,q}$ lands inside $\mathrm{W}^{p,q}$.

## 2. Geometric interpretation of Hodge structures

My goal will be to explain how both pure Hodge stuctures and the functorial association of a pure Hodge structure of a smooth projective variety can be produced geometrically from a curve.

To motivate this construction, I want to introduce a definition which is slightly less standard.

> DEFINITION 2.1. A pure complex Hodge structure of weight $n$ is a complex vector space V, along with the data of two filtrations $F^i\mathrm{V}$ and $\overline{F}^i\mathrm{V}$ such that
>
> $$\mathrm{V} = \bigoplus_{p+q=n} \mathrm{V}^{p,q},$$
>
> where $\mathrm{V}^{p,q} = F^p\mathrm{V} \cap \overline{F}^q\mathrm{V}$.

In our case, there is a natural definition for these filtrations coming from the Hodge filtration: there exists a filtration $F^p$ on $\mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}(\mathbf{C}), \mathbf{C})$ by taking $\bigoplus_{p' \geq p, p'+q'=n} \mathrm{H}^{p'}(\mathrm{X}, \Omega^{q'})$. There is an action of complex conjugation on this, giving a conjugate filtration. This defines $\overline{F}$.

Our first goal will be to geometrically encode these structures by realizing them in a category of vector bundles on a curve. Once this is done, it will be more clear how to encode the real Hodge structure attached to X.

Our starting point will be the following observation which will help encode filtrations. We let $\mathbb{G}_m$ act on $\mathbf{A}^1 = \operatorname{Spec} \mathbf{C}[x]$ via setting the action of $t \in \mathbb{G}_m$ to be

$$x \mapsto tx.$$

LEMMA 2.2. A vector bundle on $\mathbf{A}^1_{\mathbf{C}}$ which is $\mathbb{G}_m$-equivariant is equivalent to a filtered vector space.

This shouldn't be too surprising, since $\mathbb{G}_m = \operatorname{Spec} \mathbf{C}[x, x^{-1}] = \operatorname{Spec} \mathbf{C}[\mathbf{Z}]$ means its representations break into a direct sum of $\mathbf{Z}$-indexed eigenspaces. In particular, an action of this group on a quasicoherent sheaf on $\mathbf{A}^1_{\mathbf{C}}$ is equivalent to the data of a $\mathbf{Z}$-grading on the corresponding $\mathbf{C}[x]$-module.

The map in one direction is as follows. Given a vector space V and filtration $F^i\mathrm{V}$, we associate the Rees module

$$\operatorname{Rees}(\mathrm{V}, F^i\mathrm{V}) := \bigoplus_{i \in \mathbf{Z}} F^i\mathrm{V} x^{-i} \subset \mathrm{V} \otimes \mathbf{C}[x, x^{-1}].$$

We use $\deg x = 1$ to give this a grading, and this will then be a graded $\mathbf{C}[x]$ module or a quasicoherent sheaf on $\mathbf{A}^1_{\mathbf{C}}$ with a $\mathbb{G}_m$ action.

Conversely, if we are given a $\mathbb{G}_m$-equivariant vector bundle we can look at the fiber over $1 \in \mathbf{A}^1_{\mathbf{C}}$. Looking at the orders of poles of $\mathbb{G}_m$-invariant sections, we obtain a filtration on this fiber. Namely, given a module $M$ which is locally free, we look at the fiber $\mathrm{V} = M_1 := M/(x-1)M$. Using $M = \bigoplus_{i \in \mathbf{Z}} M_i$, we define

$$F^i\mathrm{V} := \operatorname{im}(M_{-i} \to \mathrm{V}).$$

This is the inverse construction.

With this, the following lemma is easy to prove.

LEMMA 2.3. Vector bundles on $\mathbf{P}^1_{\mathbf{C}}$ which are $\mathbb{G}_m$-equivariant are equivalent to vector spaces with two filtrations.

*Proof.* Basically, we have two ways to get the same vector space: look at the affine chart $\mathbf{A}^1_{\mathbf{C}} = \mathbf{P}^1_{\mathbf{C}} \setminus \infty$ or $\mathbf{A}^1_{\mathbf{C}} = \mathbf{P}^1_{\mathbf{C}} \setminus 0$. Both of these give you vector bundles on $\mathbf{A}^1_{\mathbf{C}}$ which are $\mathbb{G}_m$ equivariant, or a complex vector space with a filtration.

There need be no compatibility between these filtrations: we can take a pair of filtrations on V and glue these together. Indeed, realize both as $\mathbb{G}_m$-equivariant vector bundles on $\mathbf{A}^1_{\mathbf{C}}$, and choose $\mathbb{G}_m$-equivariant trivializations of both. We identify these on

$$\mathbf{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$$

and then use this to glue them together. □

A natural question then arises. Given a $\mathbb{G}_m$-equivariant vector bundle on $\mathbf{P}^1_{\mathbf{C}}$, what conditions ensure that it corresponds to a pure complex Hodge structure? The answer is quite simple. Let $\mathbb{G}_m$ act on $\mathbf{P}^1_{\mathbf{C}}$ via $[x : y] \mapsto [tx, y]$.

> Theorem 2.4. A $\mathbb{G}_m$-equivariant vector bundle on $\mathbf{P}^1_{\mathbf{C}}$ is a pure complex Hodge structure if and only if it is semistable after forgetting the $\mathbb{G}_m$ action.

Here, V being semistable means for all nonzero proper subbundles $\mathrm{W} \leq \mathrm{V}$ we have

$$\deg \mathrm{W}/\mathrm{rankW} \leq \deg \mathrm{V}/\mathrm{rankV}.$$

Since we know a full classification of vector bundles, we can say semistable is equivalent to being of the form $\mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(n)^{\oplus i}$. If we had different $n$, taking the largest one gives a destabilizing subbundle.

To see why this is reasonable, suppose we take $\mathcal{O}(n)$ and look at the action on $\mathcal{O}(n)|_{\mathbf{A}^1_{\mathbf{C}}} \simeq \mathcal{O}_{\mathbf{A}^1_{\mathbf{C}}}$; say the associated filtration is $\mathbf{C}$, until we get to $F^i = 0$.

On the other affine chart with coordinate $x$, since multiplication by $x^{-n}$ is the transition function, we get $\mathbf{C}$ until $\overline{F}^{i+n} = 0$. We can see this change by looking at the Rees module. This means $\mathcal{O}(n)$ will give us some pure Hodge structure of weight $n$. In general, if we mix line bundles of different degrees there will not be a uniform shift so we cannot get $\mathrm{V} = \bigoplus_{i+j=n} F^i\mathrm{V} \cap \overline{F}^j\mathrm{V}$. This is why we need semistability.

While this answers the question of geometrization for the complex Hodge structure in a satisfactory way, there is a deeper structure we want to actually capture.

Namely, the group $\mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}(\mathbf{C}), \mathbf{R})$ has what is known as a *real Hodge structure*. This is now in the realm of things people actually consider in Hodge theory.

> Definition 2.5. A pure real Hodge structure of weight $n$ is a vector space $\mathrm{V}_{\mathbf{R}}$ along with a decomposition
>
> $$\mathrm{V}_{\mathbf{C}} \simeq \bigoplus_{p+q=n} \mathrm{V}^{p,q}$$
>
> where $\mathrm{V}^{p,q} = \overline{\mathrm{V}^{q,p}}$.

Via the same filtrations, $\mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}(\mathbf{C}), \mathbf{R})$ carries a real Hodge structure. Given what we've seen with $\mathbf{P}^1_{\mathbf{C}}$, it's natural to expect that a similar thing should work but with a real form of $\mathbf{P}^1_{\mathbf{C}}$.

This is indeed the case! The twistor line $\mathbf{P}^1_{\mathrm{tw}}$ is constructed from $\mathbf{P}^1_{\mathbf{C}}$: there is an automorphism of $\mathbf{P}^1_{\mathbf{C}}$ sending $z \mapsto -\frac{1}{\bar{z}}$, which gives a descent datum.

This gives us a *real form* of $\mathbf{P}^1_{\mathbf{C}}$, denoted by $\mathbf{P}^1_{\mathrm{tw}}$, which is an algebraic curve over $\mathbf{R}$. We can explicitly write it as follows:

$$\mathbf{P}^1_{\mathrm{tw}} = \operatorname{Proj} \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2).$$

Observe that this has no real points. There is a covering map

$$\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathrm{tw}}$$

so we deduce that $\pi_1(\mathbf{P}^1_{\mathrm{tw}}) = \mathbf{Z}/2\mathbf{Z}$.

Before, we had an action on $\mathbf{P}^1_{\mathbf{C}}$ by $\mathbb{G}_m$. This action descends to an action by the unitary group $\mathrm{U}(1)/\mathbf{R}$, given by

$$\mathrm{U}(1) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, x^2 + y^2 = 1 \right\} \subseteq \mathrm{GL}_{2,\mathbf{R}}.$$

Note that the Deligne torus is different, we just ask for the determinant to be a unit. With the way $\mathbf{P}^1_{\mathrm{tw}}$ is explicitly described, you can also see the natural action. Note that the action is actually free on $\mathbf{P}^1_{\mathrm{tw}} \setminus \infty$.

If we want to be a bit more suggestive, this actually comes equipped with a natural action of the Weil group $\mathrm{W}_{\mathbf{R}} = \mathbf{C}^\times \cup j\mathbf{C}^\times \subseteq \mathbb{H}$. You can think of $\mathbf{P}^1_{\mathrm{tw}}$ as the projectivized cone of the *scheme* of traceless norm zero elements of $\mathbb{H}$ (this has $\mathbf{C}$ points), which is how we get a natural action. Note that this is really sort of false advertising though, because it really is an action of the Weil group modulo its center.

Define for half integral $\lambda$

$$\mathcal{O}_{\mathbf{P}^1_{\mathrm{tw}}}(\lambda) := \pi_* \mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(2\lambda)$$

and for integral lambda set this to be $\mathcal{L}$ such that the pullback $\pi^*\mathcal{L}$ is $\mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(2\lambda)$.

> Theorem 2.6. We can classify vector bundles on $\mathbf{P}^1_{\mathrm{tw}}$ via the classification on $\mathbf{P}^1_{\mathbf{C}}$. Namely, for every finite decreasing half-integer sequence $\{\lambda_i\}$ we attach the vector bundle $\bigoplus_i \mathcal{O}_{\mathbf{P}^1_{\mathrm{tw}}}(\lambda)$. This is a bijection.
>
> Semistable vector bundles of slope zero correspond to real vector spaces.

> REMARK 2.7. In general, G-bundles up to isomorphism are classified by $\mathrm{B}(\mathrm{G}) := \mathrm{H}^1(\mathrm{W}_{\mathbf{R}}, \mathrm{G}(\mathbf{C}))$ of equivalence classes of cocycles $\mathrm{W}_{\mathbf{R}} \to \mathrm{G}(\mathbf{C})$ whose restriction to $\mathbf{C}^\times$ is algebraic, where the Weil group acts by its restriction to $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$.

With this classification in hand, we'll try to adapt what we had for pure real Hodge structures of weight $n$.

As there is a $\mathrm{U}(1)$ action on $\mathbf{P}^1_{\mathrm{tw}}$, it makes sense to talk about $\mathrm{U}(1)$-equivariant vector bundles.

The starting point is the following result.

> LEMMA 2.8. The category of $\mathrm{U}(1)$-equivariant vector bundles on $\mathbf{P}^1_{\mathrm{tw}}$ is equivalent to the category of real vector spaces equipped with a filtration on their complexification.

*Proof.* The idea is to use Beauville-Laszlo gluing. Let me revisit $\mathbf{A}^1_{\mathbf{C}}$ to explain how it works. Basically, if we look at $\mathbb{G}_m \subset \mathbf{A}^1_{\mathbf{C}}$, the action of $\mathbb{G}_m$ restricts to this subset.

The gluing theorem tells us that in such a case, the data of a $\mathbb{G}_m$-equivariant vector bundle is equivalent to the following:

- A $\mathbb{G}_m$-equivariant vector bundle $\mathrm{V}$ on $\mathbb{G}_m$ (necessarily trivial).
- A $\mathbb{G}_m$-equivariant vector bundle $\mathrm{V}_\varepsilon$ on $\operatorname{Spec} \mathbf{C}[[x]]$.
- A gluing datum on $\operatorname{Spec} \mathbf{C}((x))$: $\mathrm{V} \otimes_{\mathbb{G}_m} \mathbf{C}((x)) \simeq \mathrm{V}_\varepsilon \otimes_{\mathbf{C}[[x]]} \mathbf{C}((x))$.

Geometrically, you can think of this as giving a vector bundle on $\mathbf{A}^1 \setminus 0$ and a small neighborhood of 0, along with a gluing datum. This works for G-equivariant vector bundles, so long as G preserves $\mathbf{A}^1 \setminus 0$.

You'll notice that this data boils down to giving a $\mathbb{G}_m$-equivariant vector bundle on $\operatorname{Spec} \mathbf{C}[[x]]$, along with a trivialization on $\mathbf{C}((x))$ as V is always trivial. Then the above data is the same as giving a lattice in $\mathrm{V} \otimes \mathbf{C}((x))$ given by $\mathrm{V}_\varepsilon$, which we just ask to be $\mathbb{G}_m$-equivariant. This is now much simpler: you'll actually just get $\sum_{i \in \mathbf{Z}} F^i \mathrm{V}[[x]] x^{-i}$, which is completely analogous to the Rees module we made earlier.

Now, we go to our case. The situation is extremely similar, except now we have a $\mathrm{U}(1)$ acton on $\mathbf{P}^1_{\mathrm{tw}}$. Pick a local coordinate $\lambda$ for the point at $\infty$, which we want to use as $\mathbf{P}^1_{\mathrm{tw}} \setminus \infty$ is stable under the $\mathrm{U}(1)$ action.

Moreover, it again can be shown $\mathrm{U}(1)$-equivariant vector bundles on $\mathbf{P}^1_{\mathrm{tw}} \setminus \infty$ are trivial. This allows us to reduce to writing down a $\mathrm{U}(1)$-equivariant vector bundle on the neighborhood $\operatorname{Spec} \mathbf{C}[[\lambda]]$ of $\infty$, or equivalently a $\mathrm{U}(1)$-equivariant lattice

$$\Lambda \subset \mathrm{V}_{\mathbf{C}} \otimes \mathbf{C}((\lambda)).$$

These are of the form $\sum_{p\in\mathbf{Z}} \lambda^{-p} F^p \mathrm{V}_{\mathbf{C}}[[\lambda]]$ and hence are equivalent to filtrations on $\mathrm{V}_{\mathbf{C}}$. Here, we should think of $\lambda$ as just a local parameter at $\infty$.

The basic difference between these examples is that in the second we went to great lengths to make V correspond to a real vector space, by working with $\mathbf{P}^1_{\mathrm{tw}} \setminus \infty$. □

In light of what happens for $\mathbf{P}^1_{\mathbf{C}}$, the following is not so surprising.

THEOREM 2.9. The category of pure real Hodge structures is equivalent to the category of U(1)-equivariant semistable vector bundles on $\mathbf{P}^1_{\mathrm{tw}}$. Weight $n$ pure real Hodge structures correspond to semistable vector bundles of slope $n/2$.

The functor here sends a pure real Hodge structure $(\mathrm{V}, \mathrm{V}^{p,q})$ to the lattice

$$\Lambda = \sum_{p\in\mathbf{Z}} \lambda^{-p} F^p \mathrm{V}[[\lambda]]$$

which lies inside $\mathrm{V} \otimes_{\mathbf{R}} \mathbf{C}((\lambda)) \simeq \mathrm{V}_{\mathbf{C}} \otimes_{\mathbf{C}} \mathbf{C}((\lambda))$. The lattice is U(1)-equivariant, and so defines a U(1)-equivariant vector bundle.

Note that a pure real Hodge structure of weight $n$ corresponds to a direct sum $\mathcal{O}_{\mathbf{P}^1_{\mathrm{tw}}}(n/2)^{\oplus i}$ where $i$ is the dimension of the underlying vector space, since we can again directly translate the meaning of semistability.

We've essentially already said how to build a pure Hodge structure out of X: you build it by modifying the vector bundle by using the Hodge filtration to give the U(1)-equivariant lattice $\Lambda$ at $\infty$. Let's just say explicitly how this is done in a slightly different way.

Consider the vector bundle

$$\mathcal{E}_{\mathrm{X}} = \mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}(\mathbf{C}), \mathbf{R}) \otimes_{\mathbf{R}} \mathcal{O}_{\mathbf{P}^1_{\mathrm{tw}}}.$$

This is semistable of slope zero, and trivial. We'll modify it to be nontrivial.

In the formal neighborhood of $\infty$ (using coordinate $\lambda$) we can identify global sections of this with the $n$th hypercohomology of

$$\Omega^0_{\mathrm{X}}[[\lambda]] \longrightarrow \Omega^1_{\mathrm{X}}[[\lambda]] \longrightarrow \Omega^2_{\mathrm{X}}[[\lambda]] \longrightarrow \ldots$$

by using the comparison with de Rham cohomology. We now make a slight modification of this: we change this to

$$\Omega^0_{\mathrm{X}}[[\lambda]] \longrightarrow \lambda^{-1}\Omega^1_{\mathrm{X}}[[\lambda]] \longrightarrow \lambda^{-2}\Omega^2_{\mathrm{X}}[[\lambda]] \longrightarrow \ldots$$

instead on $\operatorname{Spf} \mathbf{C}[[\lambda]]$. We then make a vector bundle $\mathcal{V}_{\mathrm{X}}$ which is isomorphic to $\mathcal{E}_{\mathrm{X}}$, except on the formal neighborhood where we replace it with this complex. This corresponds to the Hodge filtration as a U(1)-invariant vector bundle on $\operatorname{Spf} \mathbf{C}[[\lambda]]$, as the corresponding lattice will be $\sum_p \lambda^p F^p \mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}(\mathbf{C}), \mathbf{R})[[\lambda]]$ for $F$ the Hodge filtration: we get $F^p$ by truncating the de Rham complex. We therefore have a modification of vector bundles

$$\mathcal{E}_{\mathrm{X}} \overset{\text{Hodge}}{\dashrightarrow} \mathcal{V}_{\mathrm{X}}$$

where $\mathcal{V}$ is a $\mathrm{U}(1)$-equivariant vector bundle of slope $n/2$.

It's worth noting that looking at the trivial bundle $\mathcal{E}_{\mathrm{X}}$, we get the following "de Rham comparison theorem":

$$(\mathcal{E}_{\mathrm{X}} \otimes \mathbf{C}((\lambda)))^{\mathrm{W}_{\mathbf{R}}} \simeq \mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}),$$

which works for a general algebraic variety $\mathrm{X}/\mathbf{R}$.

In summary, there is the following dictionary:

| **Hodge theory** | **Twistors** |
|---|---|
| Real Hodge structure (filtration on $\mathrm{V}_{\mathbf{C}}$) | $\mathrm{U}(1)$-equivariant vector bundle on $\mathbf{P}^1_{\mathrm{tw}}$ |
| Pure Hodge structure | Semistable $\mathrm{U}(1)$-equivariant vector bundle on $\mathbf{P}^1_{\mathrm{tw}}$ |
| Weight $n$ of Hodge structure | Slope $n/2$ of vector bundle |
| Hodge structure of X | Modification $\mathrm{V}_{\mathrm{X}}$ of $\mathrm{W}_{\mathrm{X}}$ at $\infty$ |
| Underlying vector space | Trivial bundle $\mathrm{V}_{\mathrm{X}}\vert_{\mathbf{P}^1_{\mathrm{tw}} \setminus \infty}$ |
| Hodge filtration | Lattice $\Lambda$ for modification at $\infty$ |

## 3. Big picture

From Simpson's perspective, this formulation is really useful because it can generalize to the notion of a mixed twistor structure. This geometrically encodes the notion of a mixed Hodge structure which appears when we work for general complex varieties, and in general one can translate results about mixed Hodge structures into ones about mixed twistor structures.

However, the notion of a mixed twistor structure is a bit more flexible than that of a mixed Hodge structure. It allows for a notion of weights in some more general settings that Hodge structures do not allow, due to their more geometric nature. For example, one can use these for the theory of mixed twistor $\mathcal{D}$-modules.

From the perspective of number theory, $\mathbf{P}^1_{\mathrm{tw}}$ is interesting because its approach to Hodge theory is largely parallel to the approach to $p$-adic Hodge theory via the Fargues-Fontaine curve. In this sense, it gives a better uniform explanation of what "Hodge theory" means.

Without saying anything about the Fargues-Fontaine curve, let me explain a little bit about how the analogy goes. The Fargues-Fontaine curve has a $\mathrm{G}_K$ action, analogous to how there is a Weil group action on $\mathbf{P}^1_{\mathrm{tw}}$.

The comparison theorem

$$(\mathcal{E}_{\mathrm{X}} \otimes \mathbf{C}((\lambda)))^{\mathrm{W}_{\mathbf{R}}} \simeq \mathrm{H}^i_{\mathrm{dR}}(\mathrm{X})$$

resembles de Rham comparison in $p$-adic Hodge theory: this tells us for a $p$-adic field $K$ and smooth proper $\mathrm{X}/K$

$$(\mathrm{H}^n_{\text{ét}}(\mathrm{X}_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathrm{B}_{\mathrm{dR}})^{\mathrm{G}_K} = \mathrm{H}^n_{\mathrm{dR}}(\mathrm{X}/K).$$

If we forget the topology, we have $\mathrm{B}_{\mathrm{dR}} \simeq \mathbf{C}_p((\lambda))$.

Additionally, we also have a $\mathrm{G}_K$-equivariant modification

$$\mathrm{H}^n_{\text{ét}}(\mathrm{X}_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathcal{O}_{\mathrm{X}} \dashrightarrow \mathcal{V}_{\mathrm{X}}$$

which can be interpreted as a certain type of semilinear algebraic object that parallels a Hodge structure. These classify certain Galois representations, and studying them via the curve can give better proofs of these classification results.