SYNTOMIFICATION AND CRYSTALLINE LOCAL SYSTEMS

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ABSTRACT. Let p be a prime, and let X be a smooth p-adic formal scheme over $\operatorname{Spf}\mathcal{O}_K$ where $K/\mathbf{Q}_p$ is a finite extension. We show that reflexive sheaves on the stack $\mathrm{X}^{\mathrm{Syn}}$ are equivalent to $\mathbf{Z}_p$-lattices in crystalline local systems on the rigid generic fiber X_η, and then use this to study the essential image of the étale realization functor on the isogeny category of perfect complexes on $\mathrm{X}^{\mathrm{Syn}}$. We also show when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth and proper that $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ is equivalent to a category of admissible filtered F-isocrystals in perfect complexes.

CONTENTS

1. INTRODUCTION

Let $K/\mathbf{Q}_p$ be a finite extension with residue field k, and let $\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)$ denote the category of crystalline Galois representations in $\mathbf{Q}_p$-vector spaces of $\mathrm{G}_K := \mathrm{Gal}(\bar{K}/K)$. In [Kis06] Kisin gave a fully faithful functor

$$\mathrm{D}_{\mathfrak{S}} : \mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{G}_K) \to \mathsf{Mod}^{\varphi}_{\mathfrak{S}}$$

to the category $\mathsf{Mod}^{\varphi}_{\mathfrak{S}}$ of Breuil-Kisin modules over $\mathfrak{S} = \mathrm{W}(k)[\![u_0]\!]$ and characterized the essential image.

A prismatic variant of this result was proven by Bhatt-Scholze. Let $\mathsf{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ denote the category of prismatic F-crystals as defined in [BS23].

Theorem 1.1 (Bhatt-Scholze, [BS23])**.** *Let $K/\mathbf{Q}_p$ be a finite extension. Then there is a symmetric monoidal equivalence*

$$\mathrm{T}_{\text{ét}} : \mathsf{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{G}_K).$$

Moreover, the diagram

$$\begin{tikzcd}
\mathsf{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \arrow[rr, "\mathrm{T}_{\text{ét}}", "\sim"'] \arrow[dr, "\mathrm{ev}"] & & \mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{G}_K) \arrow[dl, "\mathrm{D}_{\mathfrak{S}}"] \\
& \mathsf{Mod}^{\varphi}_{\mathfrak{S}} &
\end{tikzcd}$$

commutes, where ev *is the evaluation functor on the Breuil-Kisin prism* $(\mathfrak{S}, \mathrm{E}(u_0))$.

Viewing this as a type of Riemann-Hilbert equivalence, it is natural to ask whether or not it admits a derived variant. In [Bha22], Bhatt-Lurie define a stack $\mathrm{X}^{\mathrm{Syn}}$ associated to a p-adic formal scheme X. Similar to previously constructed stacks such as X^{Δ} in [BL22b], a central idea is that $\mathrm{R}\Gamma(\mathrm{X}^{\mathrm{Syn}}, \mathcal{O}\{i\})$ produces the syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{Syn}}(\mathrm{X}, \mathbf{Z}_p(i))$.

Let X_η denote the rigid generic fiber of X, and define $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ to be the full subcategory of $\mathrm{D}_{\text{proét}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ consisting of locally bounded derived p-complete objects whose mod p reduction has cohomology sheaves that are locally constant with finitely generated stalks. Bhatt-Lurie construct a t-exact étale realization functor

$$\mathrm{T}_{\text{ét}} : \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \to \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)$$

utilizing the equivalence in Corollary 3.8 in [BS23].

In the case of a point, they extend the result of Bhatt-Scholze to the derived setting after rationalization by integrally lifting prismatic F-crystals to $\mathcal{O}_K^{\mathrm{Syn}}$.

Theorem 1.2 (Bhatt-Lurie, [Bha22])**.** *Let $\mathcal{E} \in \mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$. Then for all i, $\mathrm{H}^i(\mathrm{T}_{\text{ét}}(\mathcal{E}))$ is crystalline.*

Moreover, there is a subcategory of reflexive sheaves $\mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \subset \mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ on which $\mathrm{T}_{\text{ét}}$ induces an equivalence $\mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \simeq \mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{G}_K)$.

Restriction to the substack $\mathcal{O}_K^{\Delta}$ induces an equivalence

$$\mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \simeq \mathsf{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}).$$

The functor $\mathrm{T}_{\text{ét}}$ factors through this restriction, and so the equivalence is compatible with the result of Bhatt-Scholze (and hence also with [Kis06]). In [Bha22], it is speculated that this result generalizes past a point in Remark 6.6.1, which is the first main result. In what follows, as in the notation section we assume X is smooth and quasicompact.

Theorem A (Theorem 3.18)**.** *For* $\mathcal{E} \in \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ *we have*

$$\mathrm{H}^i(\mathrm{T}_{\text{ét}}(\mathcal{E}))[1/p] \in \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)$$

for all i*. Moreover, there is a subcategory of reflexive sheaves* $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \subset \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$ *(see Definition 3.1) on which* $\mathrm{T}_{\text{ét}}$ *induces an equivalence* $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$*.*

The second assertion also holds without a quasicompactness assumption (which is only introduced to control isogeny categories). In [GL23], Guo-Li already partially answered this question. By combining with the main result of Guo-Reinecke ([GR24]) they produce a fully faithful functor

$$\Pi_{\mathrm{X}} : \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta) \to \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}) = \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})^{\heartsuit}$$

such that $\mathrm{T}_{\text{ét}}(\Pi_{\mathrm{X}}(\mathcal{L})) \simeq \mathcal{L}$. What remains is to characterize the essential image in a way analogous to the case of a point. We show the essential image of this functor is $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}})$, the subcategory of reflexive objects in $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$. This alternate description easily implies the first part of the theorem, as well as the following corollary deduced by understanding $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$. In what follows, the same remark about how we interpret isogeny categories applies when X is not quasicompact.

Corollary 1.3 (Corollary 4.1)**.** *The* t*-exact functor*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \xrightarrow{\mathrm{T}_{\text{ét}}[1/p]} \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$$

induces an equivalence $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p] \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)$ *on the heart. The essential image of* $\mathrm{T}_{\text{ét}}[1/p]$ *contains the essential image of*

$$\mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)) \to \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p],$$

and is contained in the full subcategory of $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ *where every cohomology sheaf is crystalline.*

In fact, all these inclusions are in general strict inclusions as explained in Remark 4.2. This corollary allows us to deduce a variant of the $\mathrm{C}_{\mathrm{cris}}$ conjecture later in Corollary 4.6. In future work, the author aims to understand the essential image of $\mathrm{T}_{\text{ét}}$ on coherent F-gauges without inverting p.

It is desirable to have a more explicit description of the category $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$. In the case of $\mathrm{X} = \mathrm{Spf}\,\mathbf{Z}_p$, it is shown in [Hau24] that

$$\mathsf{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_{\mathbf{Q}_p})).$$

One may also interpret this, using Colmez-Fontaine's equivalence $\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_{\mathbf{Q}_p}) \simeq \mathrm{MF}^{\varphi,\mathrm{wa}}_{\mathbf{Q}_p}$ in [CF00] between crystalline Galois representations and filtered F-isocrystals, as an equivalence $\mathsf{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathrm{MF}^{\varphi,\mathrm{wa}}_{\mathbf{Q}_p})$. This statement is what generalizes; it is the second main result.

Theorem B (Theorem 4.23)**.** *Assume that* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *is smooth proper. Then there is an equivalence of categories*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \simeq \mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X}).$$

Here $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ *is as in Definition 4.17, and the superscript* adm *denotes the full subcategory of objects with admissible cohomologies (as in Definition 4.18).*

While we initially define $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ using formal stacks, we show there is an explicit description of this category using D-modules. One might be tempted to generalize Hauck's result to general X by trying to relate $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ to $\mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta))$; we show that in Example 4.7 this already fails for $\mathbf{P}^1$ due to a failure of full faithfulness and we expect that it rarely works outside of the case of $\operatorname{Spf}\mathcal{O}_K$, where we confirm it indeed generalizes this way (Proposition 4.14). As a positive result in this direction, in the category $\mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta))$ the RHom would be computed by crystalline extension groups. The following result shows this still holds in a limited sense.

Proposition 1.4 (Proposition 4.3)**.** *Assume* X *is smooth and quasicompact over* $\operatorname{Spf}\mathcal{O}_K$ *and let* $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$*. Then the étale realization induces an isomorphism*

$$\mathrm{H}^1_{\mathrm{Syn}}(\mathrm{X}, \mathcal{E})[1/p] \simeq \mathrm{Ext}^1_{\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)}(\mathbf{Q}_p, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]).$$

In particular, if $\mathrm{X} = \operatorname{Spf}\mathcal{O}_K$ *then* $\mathrm{H}^1_{\mathrm{Syn}}(\mathrm{X}, \mathcal{E})[1/p] = \mathrm{H}^1_f(\mathrm{G}_K, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p])$ *where* H^1_f *denotes the Bloch-Kato Selmer group.*

Overview of the proofs. To make sense of our results, we first need to set up some general theory about t-structures on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ and coherent sheaves which is done in §2.2. For Theorem A, we first show the second part of the theorem that $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$. The first step in this argument is to analyze the kernel of the étale realization in §3.1. We first show that prismatic F-crystals in the kernel of the étale realization are p-torsion. The main difficulty in deducing the claimed kernel of the étale realization in Theorem 3.5 is showing the same claim for F-gauges, namely that F-gauges in the kernel of $\mathrm{T}_{\text{ét}}$ are also p-torsion. This is done by adapting the method of [Bha22] by constructing a graded connection on the Nygaard associated graded. We then show that Guo-Reinecke's category $\mathsf{Vect}^{\mathrm{an},\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ can be viewed as reflexive objects in coherent prismatic F-crystals (using §3.1). Using Guo-Li's functor Π_{X} from [GL23] to lift these to sheaves on $\mathrm{X}^{\mathrm{Syn}}$, we then use §3.1 again to show the output is reflexive (one must also verify coherence first, as their definition differs and does not use the stack). We can then deduce the first part of Theorem A from the second part.

For Theorem B, we use Theorem A to deduce Corollary 4.1 which is used to deduce the essential image in Theorem B (by using the statement about the heart). Full faithfulness is the main difficulty, which is shown using a variant of the Beilinson fiber square over $\mathcal{O}_K$ with coefficients (see [AMMN22]). The key input for this is a cohomology calculation on $\mathcal{O}_K^{\mathrm{Syn}}$ using the methodology of [AKN24]. Similar squares appear in [Hau24] and [Dev25], but restrict either to $\mathrm{X}/\operatorname{Spf}\mathbf{Z}_p$ smooth or work relative to the q-de Rham prism instead of the Breuil-Kisin prism and so cannot be used to deduce Theorem B for smooth proper $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$.

Notation. Throughout, we fix a finite extension $K/\mathbf{Q}_p$ with residue field k and ring of integers $\mathcal{O}_K$. We write X for a smooth and quasicompact p-adic formal scheme over $\operatorname{Spf}\mathcal{O}_K$, and $\mathrm{X}_\eta/\operatorname{Spa}K$ for its rigid generic fiber. We will adopt the conventions of [Bha22] for the various stacks $\mathrm{X}^{\mathrm{Syn}}$, $\mathrm{X}^{\mathrm{Nyg}}$, $\mathrm{X}^{\mathbb{\Delta}}$, etc. We write $\mathrm{X}_{\mathbb{\Delta}}$ for the absolute prismatic site of X, and $\mathcal{O}_{\mathbb{\Delta}}$ for its structure sheaf. We use $(-)^{\wedge}_{\mathrm{I}}$ to denote derived I-adic completion (and similarly for p). We also use the slightly nonstandard term “flat-local surjection" in the sense of Definition 2.4 to mean a map which is locally surjective in the fpqc topology; this notion suffices for the descent results we need. All categories are regarded as $(\infty,1)$ categories, although these will not play an essential role in the paper. We use cohomological shift conventions everywhere.

For a smooth p-adic formal scheme $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$, we set $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ to be the category of pro-étale crystalline $\widehat{\mathbf{Z}}_p$-local systems as in Definition 2.31 of [GR24]. We write $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)$ for the isogeny category $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)[1/p]$ (so in particular they always admit a lattice), and in general use $(-)[1/p]$ when applied to a $\mathbf{Z}_p$-linear category to denote its isogeny category.

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2. Preliminaries

2.1. **Stacks.** In [Dri24] Drinfeld introduced the formal stack[1] $\mathbf{Z}_p^{\mathbb{\Delta}}$, and later in [BL22b] this was extended to the relative prismatization $(\mathrm{X}/\mathrm{A})^{\mathbb{\Delta}}$ and the absolute prismatization $\mathrm{X}^{\mathbb{\Delta}}$ for p-adic formal schemes X. As we will only ever need p-adic formal schemes which are at worst quasisyntomic, we will make definitions in this generality.

We first recall the definition of the absolute prismatization, using the definition of a Cartier-Witt divisor ([BL22a] Definition 3.1.4). The functor assigning a p-nilpotent ring R to the groupoid of Cartier-Witt divisors gives a stack $\mathbf{Z}_p^{\mathbb{\Delta}}$, and the construction assigning a Cartier-Witt divisor $\alpha:\mathrm{I}\to\mathrm{W}(\mathrm{R})$ to the derived quotient $\overline{\mathrm{W}(\mathrm{R})}=\mathrm{W}(\mathrm{R})/\mathrm{I}$ gives a stack $\mathbf{G}_a^{\mathbb{\Delta}}\to\mathbf{Z}_p^{\mathbb{\Delta}}$. Carrying out transmutation, one arrives at the following definition.

Definition 2.1 ([BL22b], Definition 3.1). *Let X be a quasisyntomic p-adic formal scheme. The absolute prismatization $\mathrm{X}^{\mathbb{\Delta}}$ is the formal stack sending a p-nilpotent ring R to the groupoid of pairs $(\alpha:\mathrm{I}\to\mathrm{W}(\mathrm{R}),\eta:\operatorname{Spec}\overline{\mathrm{W}(\mathrm{R})}\to\mathrm{X})$ where α is a Cartier-Witt divisor.*

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[1] By a *formal stack* X we will always mean an accessible sheaf $\mathrm{R}\mapsto\mathrm{X}(\mathrm{R})$ for flat (fpqc) topology on p-nilpotent rings valued in groupoids.

In this generality, as shown in [BL22b] Theorem 6.4 we know that $\mathrm{R\Gamma}(\mathrm{X}^{\Delta}, \mathcal{O})$ computes the prismatic cohomology of X. We note also that Theorem 6.5 of loc. cit. shows that $\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\Delta})$ is the category of prismatic crystals in quasicoherent sheaves.

In [Bha22], the additional stacks $\mathrm{X}^{\mathrm{Nyg}}$ and $\mathrm{X}^{\mathrm{Syn}}$ are defined, which have similar properties. Namely, following Drinfeld in [Dri24] the notion of a *filtered Cartier-Witt divisor* is introduced in [Bha22] Definition 5.3.1. Analogously, $\mathbf{Z}_p^{\mathrm{Nyg}}$ sends a p-nilpotent ring R to the groupoid of filtered Cartier-Witt divisors $d : \mathrm{M} \to \mathrm{W}$ over R. The construction sending a filtered Cartier-Witt divisor $d : \mathrm{M} \to \mathrm{W}$ to

$$(\mathrm{W}/\mathrm{M})(\mathrm{R}) := \mathrm{R\Gamma}(\mathrm{Spec}\,\mathrm{R}, \mathrm{W}/\mathrm{M})$$

produces an animated W-algebra stack $\mathbf{G}_a^{\mathrm{Nyg}} \to \mathbf{Z}_p^{\mathrm{Nyg}}$. Transmutation provides the following general definition.

Definition 2.2 ([Bha22], Definition 5.3.10)**.** *Let* X *be a quasisyntomic* p*-adic formal scheme. The absolute Nygaard-filtered prismatization* $\mathrm{X}^{\mathrm{Nyg}}$ *is the formal stack sending a* p*-nilpotent ring* R *to the groupoid of pairs* $(d : \mathrm{M} \to \mathrm{W}, \eta : \mathrm{Spec}(\mathrm{W}/\mathrm{M})(\mathrm{R}) \to \mathrm{X})$ *where* d *is a filtered Cartier-Witt divisor.*

The stack $\mathrm{X}^{\mathrm{Nyg}}$ comes with two maps

$$j_{\mathrm{dR}}, j_{\mathrm{HT}} : \mathrm{X}^{\Delta} \to \mathrm{X}^{\mathrm{Nyg}}$$

defined via transmutation from [Bha22] Constructions 5.3.5 and 5.3.2.

Finally, we will primarily be dealing with the stack $\mathrm{X}^{\mathrm{Syn}}$. In [BL22a] Construction 7.4.1, the definition of syntomic cohomology $\mathrm{R\Gamma}_{\mathrm{Syn}}(\mathrm{X}, \mathbf{Z}_p(i))$ is given as the fiber

$$\varphi\{i\} - \mathrm{can} : \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_{\mathrm{X}}\{i\} \to \Delta_{\mathrm{X}}\{i\}.$$

Here, $\varphi\{i\}$ is the ith divided Frobenius. This construction can be realized in the stacky perspective by defining $\mathrm{X}^{\mathrm{Syn}}$ as the pushout in formal stacks

$$\begin{tikzcd}
\mathrm{X}^{\Delta} \sqcup \mathrm{X}^{\Delta} \arrow[r, "j_{\mathrm{dR}} \sqcup j_{\mathrm{HT}}"] \arrow[d] & \mathrm{X}^{\mathrm{Nyg}} \arrow[d, "j_{\mathrm{Nyg}}"] \\
\mathrm{X}^{\Delta} \arrow[r, "j_{\Delta}"] & \mathrm{X}^{\mathrm{Syn}}
\end{tikzcd}$$

as in [Bha22] Definition 6.1.1. This pushout is in particular a coequalizer. We will use several times that the map $\mathrm{X}^{\mathrm{Nyg}} \to \mathrm{X}^{\mathrm{Syn}}$ is étale.

The map j_{dR} corresponds to can, while j_{HT} corresponds to φ. One may deduce that

$$\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}}) = \mathrm{eq}(\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Nyg}}) \underset{j^*_{\mathrm{HT}}}{\overset{j^*_{\mathrm{dR}}}{\rightrightarrows}} \mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\Delta}))\ ,$$

and in particular on cohomology we get a natural exact triangle

$$\mathrm{R\Gamma}(\mathrm{X}^{\mathrm{Syn}}, \mathcal{E}) \longrightarrow \mathrm{R\Gamma}(\mathrm{X}^{\mathrm{Nyg}}, j^*_{\mathrm{Nyg}}\mathcal{E}) \xrightarrow{j^*_{\mathrm{HT}} - j^*_{\mathrm{dR}}} \mathrm{R\Gamma}(\mathrm{X}^{\Delta}, j^*_{\Delta}\mathcal{E})$$

where j_{Nyg} and j_{Δ} are as in the pushout square defining $\mathrm{X}^{\mathrm{Syn}}$. Using Example 6.1.8 in [Bha22], putting $\mathcal{E} = \mathcal{O}\{i\}$ we recover the definition in [BL22a].

As pointed out in Remark 6.3.4 in [Bha22], the category $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ admits a restriction functor yielding perfect prismatic F-crystals. We use $\mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ to refer to prismatic F-crystals in perfect complexes as defined in [GL23] §3.1. We will often use the following observation from [Bha22].

Lemma 2.3 (Remark 6.3.4 in [Bha22])**.** *Restriction to* X^{Δ} *gives a functor*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \to \mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}).$$

This will be useful for us since the construction of the étale realization factors through this restriction functor, so many arguments may be reduced to arguments about prismatic F-crystals.

2.2. **Coherent sheaves.** We will also need a theory of coherent sheaves for $\mathrm{X}^{\mathrm{Syn}}$ when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth. One possible definition is given in [GL23], but it is likely not the same as the natural category to associate using the stacky approach to F-gauges. In this subsection we will use an approach which is closer to how [GL23] treats defining coherent prismatic F-crystals but adapted to $\mathrm{X}^{\mathrm{Nyg}}$ and $\mathrm{X}^{\mathrm{Syn}}$.

We will first show that when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth that $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ comes equipped with a natural t-structure. To do so, we will need to use flat-local surjections. However, we note that we do not mean a flat covering of stacks. Rather, we use the following slightly weaker notion which suffices for descent.

Definition 2.4. *Let* $\mathrm{X} \to \mathrm{Y}$ *be a map of* p*-adic formal stacks in the flat topology. Then we say* $\mathrm{X} \to \mathrm{Y}$ *is a flat-local surjection if it is surjective locally in the flat topology.*

This differs from the usual notion of a *flat covering of* p*-adic formal stacks*, which would ask that the map is representable and the pullback along $\operatorname{Spec}\mathrm{R} \to \mathrm{Y}$ for a p-nilpotent ring R induces a flat covering of $\operatorname{Spec}\mathrm{R}$. We will distinguish this second stronger notion by calling the map a flat cover of formal stacks.

This notion is sufficient for most results. Indeed, it suffices if we want to deduce descent for $\mathrm{D}_{\mathrm{qc}}(-)$, $\mathsf{Perf}(-)$, and $\mathsf{Vect}(-)$. This property is sufficient to deduce that $\mathrm{X} \to \mathrm{Y}$ is an effective epimorphism in the topos of flat sheaves, which means $\mathrm{Y} \simeq \operatorname{colim} \mathrm{X}^{\times_{\mathrm{Y}}(-)}$ where $\mathrm{X}^{\times_{\mathrm{Y}}(-)}$ are the terms of the Čech nerve. See [HP24] Appendix B for a proof of this result. Since $\mathrm{D}_{\mathrm{qc}}(-)$, Perf and $\mathsf{Vect}(-)$ are sheaves for the flat topology, the desired claim follows. Here we use that finite presentation is a flat local condition.

Lemma 2.5 ([BL22b] Lemma 6.3)**.** *Let* $\mathrm{X} \to \mathrm{Y}$ *be a quasisyntomic cover of quasisyntomic* p*-adic formal schemes. Then* $\mathrm{X}^{\Delta} \to \mathrm{Y}^{\Delta}$ *is surjective locally in the flat topology.*

Remark 2.6. One can upgrade this claim to $X^{\Delta} \to Y^{\Delta}$ being a flat covering of formal stacks as in Proposition 2.19. We will largely not need this, as for many arguments the notion of a flat-local surjection is easier and suffices.

To define a t-structure on $\mathsf{Perf}(X^{\mathrm{Syn}})$, we will pick Rees stacks as flat-local surjections and show that the t-structure is independent of the choice we made.

Definition 2.7. *Let* (A, I) *be a prism. The completed Rees stack* $\mathrm{Rees}_{I^\bullet}(A)$ *is defined as the stack*

$$\mathrm{Spf}(\bigoplus_{i\in\mathbf{Z}}(\mathrm{Fil}^i_{I^\bullet} A)t^{-i})^{\wedge}_{(p,I)}/\mathbf{G}_m$$

where t *is given degree 1. We endow the graded ring* $(\bigoplus_{i\in\mathbf{Z}}(\mathrm{Fil}^i_{I^\bullet} A)t^{-i})^{\wedge}_{(p,I)}$ *with the* (p, I)*-adic topology.*

When $I = (d)$, we have

$$\mathrm{Rees}_{I^\bullet}(A) \simeq (\mathrm{Spf}\, A\langle u, t\rangle/(ut - d))/\mathbf{G}_m$$

where $\deg(t) = 1, \deg(u) = -1$.

Analogously to how [GL23] defines coherent prismatic F-crystals, we use Breuil-Kisin prisms to construct coherent F-gauges.

Proposition 2.8 ([GL23] Proposition 3.7, Corollary 3.6). *Let* $X/\,\mathrm{Spf}\,\mathcal{O}_K$ *be a smooth affine formal scheme. Let* $\tilde{R}$ *be a choice of smooth lift of the special fiber* $X_s/\,\mathrm{Spec}\,k$ *to* $W(k)$ *and* $E(u_0)$ *an Eisenstein polynomial for a uniformizer* $\pi \in \mathcal{O}_K$*. Then there is a prism*

$$(A, I) \simeq (\tilde{R}[\![u_0]\!], E(u_0))$$

such that $\mathrm{Spf}\, A/I \simeq X$*, and* (A, I) *covers the final object* $* \in \mathsf{Sh}(X_{\Delta})$*. Moreover the Frobenius on* A *is finite, quasisyntomic, and faithfully flat.*

Given such a prism (A, I), there is a natural cover we can produce. In what follows, $A^{(1)}$ is the Frobenius twist $A \otimes_{A,\varphi} A$ with the relative Frobenius sending $a \otimes b \mapsto \varphi(a)b$. We also use the symbol (1) to denote the Frobenius twist of prismatic cohomology when applicable, as well as φ^* for clarity in some situations.

Proposition 2.9. *Assume that* X *is a smooth affine formal scheme over* $\mathrm{Spf}\,\mathcal{O}_K$*, and* (A, I) *is a Breuil-Kisin prism such that* $X \simeq \mathrm{Spf}(A/I)$*. Then*

$$\rho_A : \mathrm{Rees}_{I^\bullet}(A) \to X^{\mathrm{Nyg}}$$

is a flat-local surjection. There is also a flat-local surjection

$$\rho'_A : \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(A^{(1)}) \to X^{\mathrm{Nyg}}$$

where the Nygaard filtration is defined as $\mathrm{Fil}^i_{\mathrm{Nyg}} = \{x \in A^{(1)} : \varphi(x) \in I^i A\}$.

Proof. This is Remark 5.5.19 in [Bha22] for the first item. For the second item there is a map

$$\varphi : \mathrm{Rees}_{\mathrm{I}^\bullet}(\mathrm{A}) \to \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(1)})$$

induced by $\varphi : \mathrm{A}^{(1)} \to \mathrm{A}$ (which is faithfully flat in this case). In fact the Rees algebra in the target is simply the base change $\mathrm{A} \otimes_\varphi \mathrm{A}\langle u, t\rangle/(ut - d)$ if $\mathrm{I} = (d)$. The map ρ_{A} will then factor as $\rho'_{\mathrm{A}} \circ \varphi$ – in the quasiregular semiperfectoid case this is easy to see (by virtue of the computation of $\mathrm{R}^{\mathrm{Nyg}}$ for R quasiregular semiperfectoid in [Bha22] as $\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \Delta_{\mathrm{R}}$, where the Nygaard filtration on Δ_{R} has $\mathrm{Fil}^i = \{x \in \Delta_{\mathrm{R}} : \varphi(x) \in \mathrm{I}^i \Delta_{\mathrm{R}}\}$ where I is the prismatic ideal of Δ_{R}), and we can deduce the general factorization from this by quasisyntomic descent and computing Čech nerves. ☐

In general, one may use the construction of Remark 5.5.19 in [Bha22] to give a map $\rho_{\mathrm{A}} : \mathrm{Rees}_{\mathrm{I}^\bullet}(\mathrm{A}) \to \mathrm{X}^{\mathrm{Nyg}}$ when $\mathrm{X}/\operatorname{Spf} \mathrm{A}/\mathrm{I}$ and I is principal. We may perform the same factorization to also produce $\rho'_{\mathrm{A}} : \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(1)}) \to \mathrm{X}^{\mathrm{Nyg}}$, and alternatively one may also write down an explicit Cartier-Witt divisor as in [LM25] Construction 1.1.6.

We may produce relative Nygaard stacks for $\mathrm{X}/\operatorname{Spf}(\mathrm{A}/\mathrm{I})$ for a prism (A, I) with I principal, which we define via the Cartesian square

$$\begin{tikzcd}
(\mathrm{X}/\mathrm{A})^{\mathrm{Nyg}} \arrow[r] \arrow[d, "\pi"] & \mathrm{X}^{\mathrm{Nyg}} \arrow[d] \\
\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \mathrm{A}^{(1)} \arrow[r, "\rho'_{\mathrm{A}}"] & (\mathrm{A}/\mathrm{I})^{\mathrm{Nyg}}
\end{tikzcd}$$

for a prism $(\mathrm{A}, \mathrm{I}) \in \mathrm{X}_{\Delta}$. This is generalized to work relative to δ pairs in [LM25] Remark 1.1.10. The cohomology of this stack computes Nygaard filtered relative prismatic cohomology via applying $\mathrm{R}\pi_*$.

When $\mathrm{X} = \operatorname{Spf} \mathrm{A}/\mathrm{I} = \operatorname{Spf} \bar{\mathrm{A}}$, the syntomic stack of X relative to A appears as a pushout

$$\begin{tikzcd}
\operatorname{Spf} \mathrm{A}^{(1)} \sqcup \operatorname{Spf} \mathrm{A}^{(1)} \arrow[r, "j_{\mathrm{dR}} \sqcup j_{\mathrm{HT}}"] \arrow[d] & \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(1)}) \arrow[d, "j_{\mathrm{Nyg}}"] \\
\operatorname{Spf} \mathrm{A}^{(1)} \arrow[r, "j_{\Delta}"] & (\bar{\mathrm{A}}/\mathrm{A})^{\mathrm{Syn}}
\end{tikzcd}$$

The map j_{HT} is given by $\operatorname{Spf} \mathrm{A}^{(1)} \to \operatorname{Spf} \mathrm{A} \simeq \left(\bigoplus_{i \in \mathbf{Z}} \mathrm{I}^i t^{-i}\right)/\mathbf{G}_m \to \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(1)})$, where the first map is induced by extension of scalars, the second map is induced by the filtered Frobenius. The map j_{dR} is the inclusion of the $t \neq 0$ locus. This pins down the general definition of the relative syntomic stack as the pushout of formal stacks

$$\begin{tikzcd}
\varphi^*(\mathrm{X}/\mathrm{A})^{\Delta} \sqcup \varphi^*(\mathrm{X}/\mathrm{A})^{\Delta} \arrow[r, "j_{\mathrm{dR}} \sqcup j_{\mathrm{HT}}"] \arrow[d] & (\mathrm{X}/\mathrm{A})^{\mathrm{Nyg}} \arrow[d, "j_{\mathrm{Nyg}}"] \\
\varphi^*(\mathrm{X}/\mathrm{A})^{\Delta} \arrow[r, "j_{\Delta}"] & (\mathrm{X}/\mathrm{A})^{\mathrm{Syn}}
\end{tikzcd}$$

where the morphisms $j_{\mathrm{dR}}, j_{\mathrm{HT}}$ of relative stacks are induced by the morphisms of absolute stacks and the morphisms from the previous square. Here, we use φ^* to mean that as a stack over $\operatorname{Spf} \mathrm{A}$ we perform base change along $\operatorname{Spf} \mathrm{A}^{(1)} \to \operatorname{Spf} \mathrm{A}$. This construction is a stacky variant of the relative syntomic cohomology in [AKN23].

Lemma 2.10. *Suppose that $\Delta_{\mathrm{X/A}}$ is discrete. Then we have a canonical identification*

$$(\mathrm{X/A})^{\mathrm{Nyg}} = \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \Delta^{(1)}_{\mathrm{X/A}}.$$

Proof. This is essentially an easier version of an argument identical to [Bha22] Theorem 5.5.10, so we will omit some details in what follows. Since the prismatic cohomology is discrete one may produce a map of stacks

$$\eta : (\mathrm{X/A})^{\mathrm{Nyg}} \to \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \Delta^{(1)}_{\mathrm{X/A}},$$

exactly as in step 3 of the proof. We have a structure map $\pi : (\mathrm{X/A})^{\mathrm{Nyg}} \to \mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \mathrm{A}^{(1)}$. Using the description of cohomology on this stack (again by a similar argument as in [Bha22] Theorem 5.5.10 Step 2) we see for $\mathrm{X}/\operatorname{Spf} \mathrm{A/I}$ smooth that $\mathrm{R}\pi_* \mathcal{O}_{(\mathrm{X/A})^{\mathrm{Nyg}}}$ is identified with Nygaard filtered prismatic cohomology under the Rees dictionary as a sheaf on $\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \mathrm{A}^{(1)}$. After Kan extension to the general case of a bounded p-adic formal scheme, we get a map $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \Delta^{(1)}_{\mathrm{X/A}}) \to \mathrm{R}\pi_* \mathcal{O}_{(\mathrm{X/A})^{\mathrm{Nyg}}}$ (as commutative algebras in $\mathrm{D}_{\mathrm{qc}}(\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}} \mathrm{A}^{(1)})$). Specializing to our X the prismatic cohomology is discrete, and then by adjunction we get η.

Then stratifying the target by the $\{t \neq 0\}$, $\{t = 0, u \neq 0\}$, and $\{t = u = 0\}$ loci we may use Theorem 7.17 of [BL22b] to show this map is an isomorphism on each stratum, hence an isomorphism via the same method of [Bha22] 5.5.10. Note that the final identification for the Hodge stack is not automatic from this result for the prismatization, but still follows from a similar argument. □

These relative stacks allow us to give a description of $\mathcal{O}_K^{\mathrm{Nyg}}$ analogous to the description via quasisyntomic descent from quasiregular semiperfectoid rings, but importantly where the first Rees stack in the equivalence is Noetherian regular (see the footnote of Remark 5.5.19 in [Bha22]; in general all the Rees charts we use are regular by the same argument). We will use this later in §4 to work with coefficients in the relative setting easily.

Corollary 2.11. *Let $(\mathrm{A}, \mathrm{I}) = (\mathrm{W}(k)[\![u_0]\!], (\mathrm{E}(u_0)))$ be a Breuil-Kisin prism for $\mathcal{O}_K$. Then there is an equivalence of stacks*

$$\mathcal{O}_K^{\mathrm{Nyg}} \simeq \operatorname{colim}_{[n] \in \Delta^{\mathrm{op}}} (\mathcal{O}_K / \mathrm{W}(k)[\![u_0, \ldots, u_n]\!])^{\mathrm{Nyg}}.$$

In particular, we get an equivalence of categories

$$\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Nyg}}) \simeq \lim_{[n] \in \Delta} \mathsf{Perf}(\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(n)}))$$

where

$$\mathrm{A}^{(n)} = \mathbbl{\Delta}^{(1)}_{\mathcal{O}_K/\mathrm{W}(k)\llbracket u_0,\ldots,u_n\rrbracket} = \varphi^*\mathrm{W}(k)\llbracket u_0,\ldots,u_n\rrbracket\left\{\frac{u_1-u_0}{\mathrm{E}(u_0)},\ldots,\frac{u_n-u_0}{\mathrm{E}(u_0)}\right\}^{\wedge}$$

with the Nygaard filtration coming from prismatic cohomology. The maps in the limit are induced by $u_i \mapsto u_j$ *(with the obvious face and degeneracy maps), and the prism structure on* $\mathrm{W}(k)\llbracket u_0,\ldots,u_n\rrbracket$ *is given by the ideal* $(\mathrm{E}(u_0))$.

Proof. Observe that as defined in the statement of the corollary we have

$$\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(n)}) \simeq (\mathcal{O}_K/\mathrm{W}(k)\llbracket u_0,\ldots,u_n\rrbracket)^{\mathrm{Nyg}},$$

using Lemma 2.10 and Theorem 9.6 in [AKN23] (due to Bhatt-Scholze), so all claims follow from showing that as stacks we have

$$\mathcal{O}_K^{\mathrm{Nyg}} \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(\mathcal{O}_K/\mathrm{W}(k)\llbracket u_0,\ldots,u_n\rrbracket)^{\mathrm{Nyg}}.$$

By Lemma 2.10 and [AKN23] we see that the right hand side can be constructed by using Nygaard filtered prismatic cohomology extended to δ-pairs; it consists of the Rees stacks corresponding to the Čech nerve for Nygaard filtered prismatic cohomology in Theorem 1.2(6). In what follows, we also use 1.2(3) to identify Nygaard filtered relative prismatic cohomology over $\mathrm{W}(k)$ with absolute Nygaard filtered prismatic cohomology.

For the perfectoid cover $\mathrm{S} = \mathrm{A}_{\mathrm{perf}}/\mathrm{I}$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(\mathrm{S}^{(n)})^{\mathrm{Nyg}} \arrow[r] & \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}\,\mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}}(\mathrm{Spf}\,\mathrm{S}^{(n)}/\mathrm{W}(k)\llbracket u_0,\ldots,u_m\rrbracket)^{\mathrm{Nyg}} \\
\mathcal{O}_K^{\mathrm{Nyg}} \arrow[u, "\sim"] \arrow[r] & \mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}}(\mathcal{O}_K/\mathrm{W}(k)\llbracket u_0,\ldots,u_m\rrbracket)^{\mathrm{Nyg}} \arrow[u, "\sim"]
\end{tikzcd}$$

where $\mathrm{Spf}\,\mathrm{S}^{(n)}$ is the nth term in the completed Čech nerve of $\mathrm{Spf}\,\mathrm{S} \to \mathrm{Spf}\,\mathcal{O}_K$. The vertical maps are equivalences since the Nygaard stack as well as $(-/\mathrm{A})^{\mathrm{Nyg}}$ send quasisyntomic covers to flat-local surjections and are compatible with Tor-independent limits: for example by the first property we may then deduce $\mathrm{S}^{\mathrm{Nyg}} \to \mathcal{O}_K^{\mathrm{Nyg}}$ is a flat-local surjection and by the second property we may identify terms of the Čech nerve with $(\mathrm{S}^{(n)})^{\mathrm{Nyg}}$ giving the left vertical equivalence. The right equivalence is similar.

It then suffices to verify the top horizontal map is an equivalence. For a quasiregular semiperfectoid $\mathrm{W}(k)\llbracket u_0\rrbracket$-algebra $\mathrm{S}^{(n)}$, we have by Theorem 1.2(6) and (3) in [AKN23] that

$$\mathrm{Fil}^\bullet_{\mathrm{Nyg}}\,\mathbbl{\Delta}_{\mathrm{S}^{(n)}} \simeq \mathrm{Fil}^\bullet_{\mathrm{Nyg}}\,\mathbbl{\Delta}^{(1)}_{\mathrm{S}^{(n)}/\mathrm{W}(k)} \simeq \lim_{m\in\Delta}\mathrm{Fil}^\bullet_{\mathrm{Nyg}}\,\mathbbl{\Delta}^{(1)}_{\mathrm{S}^{(n)}/\mathrm{W}(k)\llbracket u_0,\ldots,u_m\rrbracket}$$

as p-complete filtered rings. Fortunately, it turns out that $\mathbbl{\Delta}^{(1)}_{\mathrm{S}^{(n)}/\mathrm{W}(k)\llbracket u_0,\ldots,u_m\rrbracket}$ is actually still discrete, so the claim follows from Lemma 2.10 and the Rees equivalence. This follows by a cotangent complex computation, which is made easy by Remark 4.21 in [BMS19] (so to check p-Tor amplitude in $[-1,-1]$ we just need to check the relevant map is quasisyntomic). Using this, via the transitivity triangle we get

$$\mathrm{L}_{\mathcal{O}_K/\mathcal{O}_K[\![v_1,\ldots,v_m]\!]}\widehat{\otimes}_{\mathcal{O}_K}\mathrm{S}^{(n)} \to \mathrm{L}_{\mathrm{S}^{(n)}/\mathcal{O}_K[\![v_1,\ldots,v_m]\!]} \to \mathrm{L}_{\mathrm{S}^{(n)}/\mathcal{O}_K}$$

and thus the claim follows by checking the p-complete Tor amplitude of the leftmost and rightmost terms. Here $\mathcal{O}_K[\![v_1,\ldots,v_m]\!] := \mathrm{W}(k)[\![u_0,\ldots,u_m]\!]/\mathrm{E}(u_0)$. Thus applying the Hodge-Tate comparison the desired discreteness claim follows. ☐

Remark 2.12. One may check the ring maps $\mathrm{A} \to \mathrm{A}^{(n)}$ are flat (rather than just (p,I)-completely flat), since the source is Noetherian and both source and target are derived (p,I)-complete (Lemma 5.15 in [Bha20]). Note also we crucially use that $\mathrm{W}(k)$ is a perfect δ-ring, so this construction doesn't generalize.

We now move to defining a t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$. Analogously to how $\mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ is defined in [Bha22], we can make the following construction.

Construction. Choose a covering of X by smooth affines U_i, and for each affine choose a Breuil-Kisin prism $(\mathrm{A}_i,\mathrm{I}_i)$ for U_i.

Then there is a flat-local surjection

$$\coprod_i \mathrm{Rees}_{\mathrm{I}_i^\bullet}\mathrm{A}_i \to \mathrm{X}^{\mathrm{Nyg}} \to \mathrm{X}^{\mathrm{Syn}}.$$

The covering is a disjoint union of quotients of Noetherian formal schemes by $\mathbf{G}_m$, thus we can define $\mathsf{Coh}(\coprod_i \mathrm{Rees}_{\mathrm{I}_i^\bullet}\mathrm{A}_i)$ in the usual way. We induce a t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ from this covering by defining the ≤ 0 and ≥ 0 parts to be perfect complexes which pull back to the ≤ 0 and ≥ 0 parts of the natural t-structure on $\mathsf{Perf}(\coprod_i \mathrm{Rees}_{\mathrm{I}_i^\bullet}\mathrm{A}_i)$.

It's possible to check manually that this t-structure does not depend on the choice of a Breuil-Kisin prism using an argument similar to [GL23], making it canonical. Let (A,I) and $(\mathrm{A}',\mathrm{I}')$ be two different choices of Breuil-Kisin prisms for X, and then take (B,J) to be the product in the absolute prismatic site of X. Then using $\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}\mathrm{B}^{(1)}$ we obtain a flat-local surjection which refines both coverings using Breuil-Kisin prisms, we can argue both Breuil-Kisin prisms give the same t-structure entirely analogously to Proposition 3.11 in [GL23].

When working with formal stacks, we can also do this by instead giving a construction of a t-structure which makes no choices and then checking it agrees with the previous construction. There is a canonical t-structure on $\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}})$ for any X, by defining $\mathrm{D}_{\mathrm{qc}}^{\geq 0}(\mathrm{X}^{\mathrm{Syn}})$ to be sheaves $\mathcal{E}$ such that for every p-nilpotent ring $f : \operatorname{Spec}\mathrm{R} \to \mathrm{X}^{\mathrm{Syn}}$ we have $f^*\mathcal{E} \in \mathrm{D}_{\geq 0}(\mathrm{R})$. We define $\mathrm{D}_{\leq 0}$ via orthogonality: that is, we say $\mathcal{E} \in \mathrm{D}_{\mathrm{qc}}^{\leq 0}(\mathrm{X}^{\mathrm{Syn}})$ when $\mathrm{RHom}(\mathcal{E},\mathcal{E}') = 0$ for all $\mathcal{E}' \in \mathrm{D}_{\mathrm{qc}}^{\geq 1}(\mathrm{X}^{\mathrm{Syn}})$ (defined analogously). This definition works for $\mathrm{D}_{\mathrm{qc}}(\mathrm{Y})$ on any formal stack Y, and when Y is $\mathrm{Rees}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}}}(\mathrm{A}^{(1)})$ or $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}$ for a Noetherian regular prism A we see restricting to Perf gives the standard t-structure on Perf. For the assertion about Perf, it is crucial that we are in a Noetherian regular situation otherwise the t-structure on quasicoherent sheaves might fail to restrict to Perf.

Remark 2.13. Pullback to the cover $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}$ for a Breuil-Kisin prism so that $\mathrm{X} \simeq \mathrm{Spf}\, \mathrm{A}/\mathrm{I}$ is t-exact for the canonical t-structure on $\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}})$ by Footnote 70 in Remark 5.5.19 of [Bha22] (combined with $\mathrm{X}^{\mathrm{Nyg}} \to \mathrm{X}^{\mathrm{Syn}}$ being an étale cover).

Proposition 2.14. *Let $\mathrm{X}/\,\mathrm{Spf}\, \mathcal{O}_K$ be smooth. The t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ does not depend on the choice of Breuil-Kisin prism used for covering.*

Proof. A priori trying to restrict the t-structure on $\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}})$ to $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ might not define a t-structure since truncations or cotruncations might fail to be perfect. However perfectness can be tested after pullback along flat-local surjections by descent, so we can test this after pullback to the cover $\sqcup_i \mathrm{Rees}_{\mathrm{I}_i^\bullet}\mathrm{A}_i$ for a covering family of Breuil-Kisin prisms after noting this pullback is t-exact by Remark 2.13. But as A_i and the associated Rees stacks are Noetherian regular, truncations/cotruncations of perfect complexes are again perfect so we conclude that we get a canonical t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$. Thus this gives a well-defined t-structure. Moreover, the argument also shows it agrees with our previous construction. ☐

We can then call this the canonical t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$.

Definition 2.15. *For the canonical t-structure on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$, define $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}) := \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})^{\heartsuit}$.*

We will also frequently use that this t-structure is compatible with étale realization.

Lemma 2.16. *The restriction functor*

$$(-)|_{\mathrm{X}^{\Delta}} : \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \to \mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$$

is t-exact for the canonical t-structure on perfect prismatic F-crystals in [GL23] Lemma 3.9. Thus $\mathrm{T}_{\text{ét}}$ is t-exact.

Proof. This is by construction, as the definitions for both t-structures use Breuil-Kisin prisms for a covering of X (so we just use the same covering).

The second claim follows from Lemma 3.16 in [GL23] and that a composite of t-exact functors is t-exact. ☐

We will also need some basic claims about duals of coherent sheaves and how $\mathrm{T}_{\text{ét}}$ interacts with these.

Definition 2.17. *Let X be a smooth p-adic formal scheme over $\mathrm{Spf}\, \mathcal{O}_K$, and $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$. We define*

$$\mathcal{E}^{\vee} := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O}_{\mathrm{X}^{\mathrm{Syn}}})$$

as the $\mathcal{O}$-linear dual, using the internal Hom in coherent sheaves. We set

$$\mathcal{E}^{\vee\vee} := \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O}_{\mathrm{X}^{\mathrm{Syn}}}), \mathcal{O}_{\mathrm{X}^{\mathrm{Syn}}}).$$

In a closed monoidal category, in this setup we always have a natural map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$.

We remark that $(-)^{\vee}$ sends coherent F-gauges to coherent F-gauges, as $\mathcal{O}_K^{\mathrm{Syn}}$ and in general $\mathrm{X}^{\mathrm{Syn}}$ for smooth quasicompact $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ admits a flat-local surjection from a Noetherian regular affine formal scheme (and pullback along this cover preserves duals).

Lemma 2.18. *Let* X *be a smooth* p*-adic formal scheme over* $\operatorname{Spf}\mathcal{O}_K$*. Let* $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$*. Then* $\mathrm{T}_{\text{ét}}$ *is symmetric monoidal, and preserves the internal Hom so that* $\mathrm{T}_{\text{ét}}(\mathcal{E}^{\vee}) = \mathrm{T}_{\text{ét}}(\mathcal{E})^{\vee}$*.*

Proof. We only add hypotheses on X so that it has a good theory of coherent sheaves.

Let us factor $\mathrm{T}_{\text{ét}}$ as

$$\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}) \to \mathsf{Coh}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathrm{I}_{\mathbb{\Delta}}]_p^{\wedge})^{\varphi=1} \simeq \mathrm{D}_{\mathrm{lisse}}^{(b)}(\mathrm{X}_{\eta}, \mathbf{Z}_p)^{\heartsuit}.$$

The first map is obtained by pullback to $\mathrm{X}^{\mathbb{\Delta}}$ and then inverting $\mathrm{I}_{\mathbb{\Delta}}$. Pullback to a substack is symmetric monoidal, as is tensoring with $\mathcal{O}_{\mathbb{\Delta}}[1/\mathrm{I}_{\mathbb{\Delta}}]^{\wedge}$. The final equivalence is symmetric monoidal as well, see section 2.2 in [IKY24] (this is the underived version).

To check the claim about duals, it suffices to check this construction preserves internal Homs. The same argument reduces us to checking this for

$$\mathsf{Coh}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathrm{I}_{\mathbb{\Delta}}]_p^{\wedge})^{\varphi=1} \simeq \mathrm{D}_{\mathrm{lisse}}^{(b)}(\mathrm{X}_{\eta}, \mathbf{Z}_p)^{\heartsuit},$$

where it is straightforward (note that for representations we take the $\mathbf{Z}_p$-linear dual and then give this the natural Galois action). □

Although we will only need the weaker statement about flat-local surjections, we will show that in fact when $\mathrm{X} \to \mathrm{Y}$ is a quasisyntomic cover, $\mathrm{X}^{\mathrm{Syn}} \to \mathrm{Y}^{\mathrm{Syn}}$ is a *flat cover* in the usual sense rather than just a flat-local surjection. To be precise, in what follows we will use the term flat cover to refer to a flat-local surjection $\mathrm{X} \to \mathrm{Y}$ of formal stacks which has the additional property that for any test object $\mathrm{S} \to \mathrm{Y}$ that is the spectrum of a p-nilpotent ring the induced map $\mathrm{X} \times_{\mathrm{Y}} \mathrm{S} \to \mathrm{S}$ is flat with the source a p-nilpotent affine scheme.

The following argument is due to Kush Singhal with help from Sasha Petrov. Any mistakes are due to the author.

Proposition 2.19. *Suppose* $f : \mathrm{X} \to \mathrm{Y}$ *is a quasisyntomic cover of bounded quasisyntomic* p*-adic formal schemes. Then the induced map* $f : \mathrm{X}^{\mathbb{\Delta}} \to \mathrm{Y}^{\mathbb{\Delta}}$ *is a flat cover of formal stacks.*

Proof. Let us first note that if $f : \mathrm{X} \to \mathrm{Y}$ is a quasisyntomic cover, in light of Lemma 6.3 in [BL22b] we only need to check the map $f^{\mathbb{\Delta}}$ is flat as a morphism of stacks (so in particular representable as a morphism of stacks).

Suppose first that $\mathrm{X} = \operatorname{Spf}\mathrm{S}$ and $\mathrm{Y} = \operatorname{Spf}\mathrm{R}$ are quasiregular semiperfectoid rings such that $\mathrm{R} \to \mathrm{S}$ is a quasisyntomic cover. We want to show that the corresponding map of initial prisms $\mathbb{\Delta}_{\mathrm{R}} \to \mathbb{\Delta}_{\mathrm{S}}$ is (p, I)-completely flat, where I is the prismatic ideal of $\mathbb{\Delta}_{\mathrm{R}}$. By definition, we want to show that $\overline{\mathbb{\Delta}}_{\mathrm{R}} \to \overline{\mathbb{\Delta}}_{\mathrm{S}}$ is p-completely flat. By Lemma 2.20, it suffices to check that the map

$$\mathrm{gr}^{\mathrm{conj}}_{*}\overline{\mathbb{\Delta}}_{\mathrm{R}} \to \mathrm{gr}^{\mathrm{conj}}_{*}\overline{\mathbb{\Delta}}_{\mathrm{S}}$$

is flat, where $\mathrm{Fil}^{\mathrm{conj}}_{*}\,\overline{\mathbb{\Delta}}_{\mathrm{R}}$ is the conjugate filtration.

By the Hodge-Tate comparison, we know that

$$\mathrm{gr}^{\mathrm{conj}}_{i}\overline{\mathbb{\Delta}}_{\mathrm{R}}/p = \left(\bigwedge\nolimits^{i}_{\mathrm{R}}\mathrm{L}_{\mathrm{R}/p\big/\tilde{\mathrm{R}}/p}\right)\{-i\}[-i]$$

where $\tilde{\mathrm{R}}$ is any perfectoid ring with a map $\tilde{\mathrm{R}} \to \mathrm{R}$, and similarly for $\overline{\mathbb{\Delta}}_{\mathrm{S}}$.

By Remark 4.21 in [BMS19] (see also Lemma 4.25), $\mathrm{L}_{\mathrm{S}/\mathrm{R}}$ actually has p-complete Tor amplitude in degree -1 (as we already know it is in $[-1,0]$ since the map is quasisyntomic), and so we have an exact sequence of S/p-modules

$$0 \to (\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p) \otimes_{\mathrm{R}/p} \mathrm{S}/p \to \mathrm{L}_{\mathrm{S}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p \to \mathrm{L}_{\mathrm{S}/\mathrm{R}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p \to 0.$$

Here all three modules are flat.

Using Propositions 25.2.4.2 and 25.2.3.4 in [Lur18], we have

$$\left(\bigwedge\nolimits^{i}\mathrm{L}_{\mathrm{R}/p\big/\tilde{\mathrm{R}}/p}\right)[-i] = \mathrm{LSym}^{i}_{\mathrm{R}/p}(\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}/p[-1]) = \mathrm{Sym}^{i}_{\mathrm{R}/p}(\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p).$$

and similarly for S. Thus the claim we want to check is that $\mathrm{Sym}^{*}_{\mathrm{S}/p}(\mathrm{L}_{\mathrm{S}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p)$ is flat over $\mathrm{Sym}^{*}_{\mathrm{R}/p}(\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p)$. The previous exact sequence can be used to show that $\mathrm{Sym}^{*}_{\mathrm{S}/p}((\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p) \otimes_{\mathrm{R}/p} \mathrm{S}/p) \to \mathrm{Sym}^{*}_{\mathrm{S}/p}\,\mathrm{L}_{\mathrm{S}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p$ is flat. Indeed, flatness of the third term implies the exact sequence is universally exact by Stacks Project 058M, and must then be a filtered colimit of split exact sequences by (6) of Stacks Project 058K. This filtered colimit is constructed for any choice of a filtered colimit presentation of the third object so we may refine this so each third term in the exact sequence is finite free by Lazard's theorem. On these split exact sequences the desired map is then a polynomial algebra extension and therefore flat, and taking a filtered colimit of flat maps is flat. By faithful flatness of $\mathrm{R}/p \to \mathrm{S}/p$ we deduce the desired claim that $\mathrm{Sym}^{*}_{\mathrm{S}/p}(\mathrm{L}_{\mathrm{S}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p)$ is flat over $\mathrm{Sym}^{*}_{\mathrm{R}/p}(\mathrm{L}_{\mathrm{R}/\tilde{\mathrm{R}}}[-1] \otimes^{\mathrm{L}} \mathbf{F}_p)$. It follows that $\mathbb{\Delta}_{\mathrm{R}} \to \mathbb{\Delta}_{\mathrm{S}}$ induces a flat cover of formal stacks.

Now suppose that X is quasiregular semiperfectoid, but Y is general. We already know $\mathrm{X}^{\mathbb{\Delta}} \to \mathrm{Y}^{\mathbb{\Delta}}$ is surjective locally in the flat topology ([BL22b] Lemma 6.3), so we will aim to show the map is flat (in what follows flat will mean the morphism is representable, and then after base change along a map from a formal scheme it is flat). Since the prismatisation commutes with Tor-independent limits, we have a pull-back diagram

$$\begin{tikzcd}
(\mathrm{X} \times_{\mathrm{Y}} \mathrm{X})^{\mathbb{\Delta}} \arrow[r] \arrow[d] & \mathrm{X}^{\mathbb{\Delta}} \arrow[d] \\
\mathrm{X}^{\mathbb{\Delta}} \arrow[r] & \mathrm{Y}^{\mathbb{\Delta}}
\end{tikzcd}$$

Since $X \times_Y X$ is also quasiregular semiperfectoid by Lemma 4.30 in [BMS19], we see that the left vertical map and the upper horizontal map are flat. Now, as $X^{\Delta} \to Y^{\Delta}$ is a surjection of sheaves in the flat topology, for any test object $S \to Y^{\Delta}$ there exists a flat cover $S' \to S$ with a map $S' \to Y^{\Delta}$ factoring through X^{Δ}, i.e. we have a diagram

$$\begin{tikzcd}
\square \arrow[r] \arrow[d, "\text{flat}"] & (X \times_Y X)^{\Delta} \arrow[r, "\text{flat}"] \arrow[d, "\text{flat}"] & X^{\Delta} \arrow[d] \\
S' \arrow[r] \arrow[rr, bend right] & X^{\Delta} \arrow[r] & Y^{\Delta}
\end{tikzcd}$$

where, *a priori*, only the indicated arrows are flat covers of formal stacks. By virtue of the left square being Cartesian $\square \to S'$ is a flat cover of formal stacks, as indicated in the diagram. As both the left and right squares are Cartesian, we deduce the outer square is as well and so $\square \simeq X^{\Delta} \times_{Y^{\Delta}} S' \to S'$ is flat. This suffices to check $X^{\Delta} \times_{Y^{\Delta}} S \to S$ is flat as we may check this flat locally on the target, and since the test object was arbitrary it follows $X^{\Delta} \to Y^{\Delta}$ is a flat cover.

Next, consider the case where Y is quasiregular semiperfectoid but X is general. Let $X_\infty \to X$ be a quasisyntomic cover with X_∞ a quasiregular semiperfectoid. Then $X_\infty \to Y$ is a quasisyntomic cover with Y also quasiregular semiperfectoid. From the previous cases, we thus know that both $X_\infty^{\Delta} \to Y^{\Delta}$ and $X_\infty^{\Delta} \to X^{\Delta}$ are flat covers of formal stacks, so the same follows for $X^{\Delta} \to Y^{\Delta}$.

Finally, let us suppose that X and Y are general quasisyntomic p-adic formal schemes. Pick quasisyntomic covers $Y_\infty \to Y$ and $X_\infty \to X$ by quasiregular semiperfectoids X_∞, Y_∞. We see $X_\infty \times_Y Y_\infty$ is also a quasiregular semiperfectoid. We have a diagram as follows:

$$\begin{tikzcd}
(X_\infty \times_Y Y_\infty)^{\Delta} \arrow[r, "\text{flat}"] \arrow[d, "\text{flat}"'] & X_\infty^{\Delta} \arrow[d, "\text{flat}"] \\
(X \times_Y Y_\infty)^{\Delta} \arrow[r] \arrow[d, "\text{flat}"'] & X^{\Delta} \arrow[d] \\
Y_\infty^{\Delta} \arrow[r, "\text{flat}"'] & Y^{\Delta}
\end{tikzcd}$$

Note that all the squares are Cartesian since prismatisation commutes with Tor independent limits. By the previous cases as well as Cartesian-ness of the squares, we already know that the indicated maps are flat morphisms of formal stacks. It follows that $X^{\Delta} \to Y^{\Delta}$ is a flat morphism of formal stacks. □

We needed the following lemma in the proof of the proposition.

Lemma 2.20. *Suppose* A *is a filtered ring and* M *a filtered* A*-module, both of which are equipped with* $\mathbf{N}$*-indexed exhaustive (honest) increasing filtrations. If the associated graded module* $\mathrm{gr}^*\mathrm{M}$ *is flat over* $\mathrm{gr}^*\mathrm{A}$*, then* M *is flat over* A*.*

Proof. This follows from [Bjö79] Proposition 3.12 in Chapter 2. ☐

Corollary 2.21. *Suppose* $f : \mathrm{X} \to \mathrm{Y}$ *is a quasisyntomic cover of bounded quasisyntomic* p*-adic formal schemes. Then the induced map* $f : \mathrm{X}^{\mathrm{Nyg}} \to \mathrm{Y}^{\mathrm{Nyg}}$ *is a flat cover of formal stacks, and similarly for* $\mathrm{X}^{\mathrm{Syn}}$*.*

Proof. We can repeat the same formal argument as with the prismatization to deduce the general case once we check that the induced map from a quasisyntomic cover of quasiregular semiperfectoid rings on Rees stacks is a surjective flat morphism of formal stacks. However, since we have already checked that in this situation $\Delta_{\mathrm{R}} \to \Delta_{\mathrm{S}}$ is a (p, I)-completely flat cover, we can simply observe that we have a Cartesian square

$$\begin{tikzcd}
\mathrm{S}^{\mathrm{Nyg}} \arrow[r] \arrow[d, "\pi"] & \mathrm{R}^{\mathrm{Nyg}} \arrow[d, "\pi"] \\
\mathrm{Spf}\, \Delta_{\mathrm{S}} \arrow[r, "\text{flat}"] & \mathrm{Spf}\, \Delta_{\mathrm{R}}
\end{tikzcd}$$

which implies the top horizontal map is a (p, I)-completely flat cover of formal stacks.

The claim for $\mathrm{X}^{\mathrm{Syn}} \to \mathrm{Y}^{\mathrm{Syn}}$ follows since $\mathrm{Y}^{\mathrm{Nyg}}$ is étale over $\mathrm{Y}^{\mathrm{Syn}}$, and also that $\mathrm{X}^{\mathrm{Nyg}} \simeq \mathrm{X}^{\mathrm{Syn}} \times_{\mathrm{Y}^{\mathrm{Syn}}} \mathrm{Y}^{\mathrm{Nyg}}$. ☐

This also allows us to give a proof of the following property of $\mathrm{X}^{\mathrm{Syn}}$ in the p-adic setting, asserted for $\mathrm{X}/\operatorname{Spec} k$ in [Bha22] Warning 4.1.3 (the warning is that separatedness can fail).

Lemma 2.22. *Let* $\mathrm{X}/\operatorname{Spf} \mathbf{Z}_p$ *be a quasisyntomic* p*-adic formal scheme with affine diagonal. Then* $\mathrm{X}^{\mathrm{Syn}}$ *has affine diagonal.*

Proof. Suppose first that X is affine, equipped with $\operatorname{Spf} \mathrm{A} \to \mathrm{X}^{\Delta}$ a flat cover where $\mathrm{A} = \Delta_{\mathrm{R}}$ for a quasiregular semiperfectoid ring R (which exists if $\operatorname{Spf} \mathrm{R} \to \mathrm{X}$ is a quasisyntomic cover by [BL22b] Lemma 6.3). Then consider the diagram

$$\begin{tikzcd}
\operatorname{Spf} \mathrm{A} \times_{\mathrm{X}^{\Delta}} \operatorname{Spf} \mathrm{A} \arrow[r] \arrow[d] & \operatorname{Spf} \mathrm{A} \times_{\operatorname{Spf} \mathbf{Z}_p} \operatorname{Spf} \mathrm{A} \arrow[d] \\
\mathrm{X}^{\Delta} \arrow[r] & \mathrm{X}^{\Delta} \times_{\operatorname{Spf} \mathbf{Z}_p} \mathrm{X}^{\Delta}.
\end{tikzcd}$$

This diagram is cartesian by writing $X^{\mathbbl{\Delta}} = X^{\mathbbl{\Delta}} \times_{X^{\mathbbl{\Delta}}} X^{\mathbbl{\Delta}}$ and using that limits commute with limits. Since $\operatorname{Spf} A \to X^{\mathbbl{\Delta}}$ is a flat cover (by Proposition 2.19) and A is flat over $\mathbf{Z}_p$, all of the fiber products are underived.

For the top map in this diagram, we can simplify to get

$$\operatorname{Spf} A \times_{X^{\mathbbl{\Delta}}} \operatorname{Spf} A = (\operatorname{Spf} R \times_X \operatorname{Spf} R)^{\mathbbl{\Delta}} = \operatorname{Spf} \mathbbl{\Delta}_{\operatorname{Spf} R \times_X \operatorname{Spf} R}.$$

For the last equivalence we use that prismatization commutes with finite Tor-independent limits (Remark 3.5 in [BL22b]). Thus the top morphism is affine. It follows $X^{\mathbbl{\Delta}}$ has affine diagonal as this property is flat local on the target.

For general X, we have a quasisyntomic covering $\tilde{X} = \sqcup_i \operatorname{Spf} R_i$ of X by quasiregular semiperfectoid rings by choosing an affine covering $\sqcup_i \operatorname{Spf} A_i$ of X and picking quasiregular semiperfectoid covers of each affine. Then there is a flat covering of $X^{\mathbbl{\Delta}} \times_{\operatorname{Spf} \mathbf{Z}_p} X^{\mathbbl{\Delta}}$ by $\tilde{X}^{\mathbbl{\Delta}} \times_{\operatorname{Spf} \mathbf{Z}_p} \tilde{X}^{\mathbbl{\Delta}}$ where $\tilde{X}^{\mathbbl{\Delta}} = \sqcup_i \operatorname{Spf} \mathbbl{\Delta}_{R_i}$ as the prismatization preserves flat covers. We can then obtain the same diagram, but with $\widetilde{X^{\mathbbl{\Delta}}}$ in place of $\operatorname{Spf} A$:

$$\begin{tikzcd}
\tilde{X}^{\mathbbl{\Delta}} \times_{X^{\mathbbl{\Delta}}} \tilde{X}^{\mathbbl{\Delta}} \arrow[r] \arrow[d] & \tilde{X}^{\mathbbl{\Delta}} \times_{\operatorname{Spf} \mathbf{Z}_p} \tilde{X}^{\mathbbl{\Delta}} \arrow[d] \\
X^{\mathbbl{\Delta}} \arrow[r] & X^{\mathbbl{\Delta}} \times_{\operatorname{Spf} \mathbf{Z}_p} X^{\mathbbl{\Delta}}.
\end{tikzcd}$$

We claim the preimages $\operatorname{Spf} R_i \times_X \operatorname{Spf} R_j$ of the affines $\operatorname{Spf} R_i \times_{\operatorname{Spf} \mathbf{Z}_p} \operatorname{Spf} R_j$ covering $\tilde{X} \times_{\mathbf{Z}_p} \tilde{X}$ are affine and quasiregular semiperfectoid. If $\operatorname{Spf} R_i$ and $\operatorname{Spf} R_j$ cover $\operatorname{Spf} A_i, \operatorname{Spf} A_j \subset X$ then since X has affine diagonal we see $\operatorname{Spf} A_i \times_X \operatorname{Spf} A_j = \operatorname{Spf} A_{ij}$ for some ring A_{ij}. We then see $\operatorname{Spf} R_i \times_X \operatorname{Spf} R_j = \operatorname{Spf}(R_i' \widehat{\otimes}_{A_{ij}} R_j')$ where $R_i' = R_i \widehat{\otimes}_{A_i} A_{ij}$ and R_j' is defined analogously. This shows the result is affine; the rings R_i', R_j' are actually again quasiregular semiperfectoid (for example Lemma 1.6 in [IKY24] applies). It then follows $\operatorname{Spf} R_i \times_X \operatorname{Spf} R_j$ is quasiregular semiperfectoid, as $R_i' \widehat{\otimes}_{A_{ij}} R_j'$ is easily checked to be quasiregular semiperfectoid.

Using the analogous affine covering $\operatorname{Spf} \mathbbl{\Delta}_{R_i} \times_{\mathbf{Z}_p} \operatorname{Spf} \mathbbl{\Delta}_{R_j}$ of $\widetilde{X^{\mathbbl{\Delta}}} \times_{\operatorname{Spf} \mathbf{Z}_p} \widetilde{X^{\mathbbl{\Delta}}}$ the preimages $\operatorname{Spf} \mathbbl{\Delta}_{R_i} \times_{X^{\mathbbl{\Delta}}} \operatorname{Spf} \mathbbl{\Delta}_{R_j} \simeq \operatorname{Spf} \mathbbl{\Delta}_{\operatorname{Spf} R_i \times_X \operatorname{Spf} R_j}$ are then all quasiregular semiperfectoid (using again the compatibility of prismatization with finite limits). Thus the map of formal schemes

$$\widetilde{X^{\mathbbl{\Delta}}} \times_{X^{\mathbbl{\Delta}}} \widetilde{X^{\mathbbl{\Delta}}} \to \widetilde{X^{\mathbbl{\Delta}}} \times_{\operatorname{Spf} \mathbf{Z}_p} \widetilde{X^{\mathbbl{\Delta}}}$$

is affine, and the same argument shows $X^{\mathbbl{\Delta}}$ has affine diagonal.

A similar argument shows X^{Nyg} has affine diagonal, where we must also make the analogous argument for the Nygaard stack that the cover by the Rees stack of a quasiregular semiperfectoid is a flat cover (using Corollary 2.21). Note that $(-)^{\mathrm{Nyg}}$ also commutes with finite Tor-independent limits.

The claim for $\mathrm{X}^{\mathrm{Syn}}$ can be checked by a similar argument, or by noting that it is obtained from a stack $\mathrm{X}^{\mathrm{Nyg}}$ with affine diagonal by gluing along two open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}}$ with disjoint images (see [Bha22] Remark 5.3.6) that are affine monomorphisms. □

3. Reflexive sheaves on $\mathrm{X}^{\mathrm{Syn}}$

We can define reflexive F-gauges for general smooth $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ in a similar way to the case of $\mathrm{X} = \operatorname{Spf}\mathbf{Z}_p$ (done in [Bha22]).

Definition 3.1. *Let* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *be smooth. An* F*-gauge in* $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$ *is reflexive if the natural map* $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ *is an isomorphism. We denote this full subcategory by* $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}})$.

This definition is slightly different, so we verify it is compatible with the one in [Bha22]. That is, after picking a Breuil-Kisin prism $(\mathrm{W}(k)[\![u_0]\!], \mathrm{E}(u_0))$ we ask that the pullback along

$$p : (\operatorname{Spf}\mathrm{W}(k)[\![u_0]\!]\langle u, t\rangle/(ut - \mathrm{E}(u_0)))/\mathbf{G}_m \to \mathcal{O}_K^{\mathrm{Syn}}$$

is a reflexive sheaf after forgetting the grading.

Lemma 3.2. *Let* $\mathcal{E} \in \mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$. *Then* $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ *is an equivalence if and only if* $\mathcal{E}$ *is reflexive in the sense of* [Bha22].

Proof. We again consider the flat-local surjection

$$p : (\operatorname{Spf}\mathrm{W}(k)[\![u_0]\!]\langle u, t\rangle/(ut - \mathrm{E}(u_0)))/\mathbf{G}_m \to \mathcal{O}_K^{\mathrm{Nyg}} \to \mathcal{O}_K^{\mathrm{Syn}}.$$

Suppose first that $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an equivalence. Then applying p^*, from flatness we have $p^*(\mathcal{E}^\vee) = (p^*\mathcal{E})^\vee$. Thus, this pulls back to $p^*\mathcal{E} \simeq (p^*\mathcal{E})^{\vee\vee}$, and $\mathcal{E}$ is reflexive in the sense of [Bha22]. Conversely, suppose we are only given the data of $p^*\mathcal{E} \simeq (p^*\mathcal{E})^{\vee\vee}$. We can check equivalences after pullback by a flat-local surjection by descent, so it follows the natural map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ inducing this must be an equivalence. □

Our goal in this section will be to prove that étale realization induces an equivalence of categories $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth. We regard this as an improvement of the following result of Guo-Li, which already produces F-gauges lifting analytic prismatic F-crystals; we simply provide a characterization of the essential image. The advantage of this characterization is that it makes it clear that there is always a natural map

$$\mathcal{E} \to \mathcal{E}^{\vee\vee}$$

to the reflexive hull of an F-gauge, where the target is reflexive.

Theorem 3.3 (Guo-Li, Theorem 3.32 in [GL23])**.** *Let* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *be a smooth* p*-adic formal scheme.*

There is a functor

$$\Pi_{\mathrm{X}} : \mathsf{Coh}^{\varphi,\mathrm{I-tf}}(\mathrm{X}_{\Delta}) \to \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$$

uniquely characterized by the condition that for $S \to \mathrm{X}$ *a* p*-completely flat quasiregular semiperfectoid cover it assigns the* F*-gauge with filtration*

$$\mathrm{Fil}^\bullet(\Pi_{\mathrm{X}}(\mathcal{E})(\mathbb{\Delta}_S)) = \varphi_{\mathcal{E}}^{-1}(\mathrm{I}^\bullet\mathcal{E}(\mathbb{\Delta}_S))$$

but with the same underlying prismatic crystal and Frobenius.

Moreover, Π_{X} *is the right adjoint of* $(-)|_{\mathrm{X}^{\mathbb{\Delta}}}$*; as the counit is an equivalence by the description,* Π_{X} *is fully faithful.*

As we will be interested in F-gauges up to isogeny later (and we will need it to argue $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$), it will be useful to first determine the kernel of

$$\mathrm{T}_{\text{ét}} : \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}) \to \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)^{\heartsuit}.$$

We provide a similar characterization of the kernel as in the case of a point shown in [Bha22], namely that it consists of $(p, v_{1,\mathrm{X}})$-power torsion F-gauges.

3.1. The kernel of the étale realization.

We will want to understand the kernel of the étale realization in order to show Π_{X} induces an equivalence

$$\mathsf{Refl}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \simeq \mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}).$$

In order to state the result, we will need to understand the section $v_{1,\mathrm{X}}$ of $\mathcal{O}\{p-1\}/p$. This can be defined very generally.

Definition 3.4 ([Bha22] Construction 6.2.1)**.** *Let* X *be a quasisyntomic formal scheme. To define a class*

$$v_{1,\mathrm{X}} \in \mathrm{H}^0_{\mathrm{Syn}}(\mathrm{X}, \mathcal{O}\{p-1\}/p)$$

we can characterize it locally on quasiregular semiperfectoid rings and then check that it descends to a well-defined class.

For a quasiregular semiperfectoid ring R *with associated prism* $(\mathbb{\Delta}_{\mathrm{R}}, \mathrm{I})$*, we identify* $\mathbb{\Delta}_{\mathrm{R}}\{p-1\}/p \simeq \mathrm{I}^{-1}/p$*. We can identify* $\mathrm{H}^0(\mathrm{R}^{\mathrm{Nyg}}, \mathcal{O}\{p-1\}/p)$ *with* $\mathrm{Fil}^{p-1}_{\mathrm{Nyg}}\mathbb{\Delta}_{\mathrm{R}}\{p-1\}/p \simeq \mathrm{I}^{-1}/p$*. Untwisting, producing a section is equivalent to a nonzero map*

$$\mathrm{I}/p \to \mathrm{Fil}^{p-1}_{\mathrm{Nyg}}\mathbb{\Delta}_{\mathrm{R}}/p$$

which exists because I/p *sits inside* Fil^p*. This map defines* $v_{1,\mathrm{R}}$ *after checking the section descends to* $\mathrm{R}^{\mathrm{Syn}}$*.*

In this subsection we will prove the following theorem by adapting the strategy for $\mathrm{X} = \mathrm{Spf}\,\mathbf{Z}_p$.

Theorem 3.5. *Let* $\mathrm{X}/\,\mathrm{Spf}\,\mathcal{O}_K$ *be smooth and quasicompact. Then the kernel of* $\mathrm{T}_{\text{ét}}$ *on* $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$ *consists precisely of coherent* F*-gauges which are killed by* $(p, v_{1,\mathrm{X}})^n$ *for* $n \gg 0$*.*

Precisely, this means that the F-gauge is p-power torsion and its mod p reduction is further v_1-power torsion.

We will first show some results about coherent prismatic F-crystals to help deduce the p-power torsion part of this claim, which is the main content (the $v_{1,\mathrm{X}}$-power torsion is easy to deduce afterward). Recall that when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth there is a standard t-structure on $\mathsf{Perf}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ via [GL23] §3, so we may make the following definition.

Definition 3.6 ([GL23])**.** *A coherent prismatic F-crystal is an F-crystal in perfect complexes in the heart of the standard t-structure on $\mathsf{Perf}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$. We denote the category of coherent prismatic F-crystals by $\mathsf{Coh}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$.*

Recall restriction of a coherent F-gauge to $\mathrm{X}^{\mathbb{\Delta}}$ naturally acquires a coherent F-crystal structure (as implied by Lemma 2.16). Despite the name, coherent prismatic F-crystals turn out to have strict restrictions on possible torsion, and in fact are vector bundles after inverting p.

Proposition 3.7. *If X is smooth and quasicompact over $\operatorname{Spf}\mathcal{O}_K$ and $\mathcal{E} \in \mathsf{Coh}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}})$, the underlying prismatic crystal satisfies $\mathcal{E}[1/p] \in \mathsf{Vect}(\mathrm{X}^{\mathbb{\Delta}})[1/p]$.*

Proof. We may test this locally for $\mathrm{X} = \operatorname{Spf}\mathrm{R}$ small affine in the sense of [DLMS24] (here we use quasicompactness to ensure we only have finite powers of p in denominators), and therefore can choose a Breuil-Kisin prism (A, I) so that $\mathrm{X} = \operatorname{Spf}\mathrm{R}$. This then gives a flat-local surjection $\operatorname{Spf}\mathrm{A} \to \mathrm{X}^{\mathbb{\Delta}}$, so we may test the property $\mathcal{E}[1/p] \in \mathsf{Vect}(\mathrm{X}^{\mathbb{\Delta}})[1/p]$ after evaluation on the Breuil-Kisin prism (A, I). By definition the A-module $\mathrm{M} := \mathcal{E}(\mathrm{A}, \mathrm{I})$ is a finite height φ-module over A, so then Proposition 4.13 in [DLMS24] applies and we deduce $\mathrm{M}[1/p]$ is a vector bundle (we remark the argument does not require a torsionfree hypothesis). ☐

3.1.1. *The locally free locus in the cyclotomic case.* In the cyclotomic case it is possible to show a stronger result about the locus where a coherent prismatic crystal (without the Frobenius isogeny condition) is a vector bundle, which we now explain. If the reader only wishes to understand the proof of Theorem 3.5, they should skip to Proposition 3.11.

We will need some preliminaries about the q-connection that evaluation on a cyclotomic prism produces. If R is p-completely étale over $\operatorname{Spf}\mathrm{W}(k)\langle \mathrm{T}_1^{\pm}, \ldots, \mathrm{T}_\ell^{\pm}\rangle$ for $n \geq 0$, then as in [Liu25] §4.1 we can produce a variant of the q-de Rham prism $\mathrm{A} = (\mathrm{R}[\![q-1]\!], [p]_{q^{p^n}}) = (\mathrm{R}[\![q-1]\!], \varphi^n([p]_q))$ as well as a q-connection on the evaluation on this prism. In the situation X_{W} is further affine and p-completely étale over a torus, we can take $\mathrm{A} = \Gamma(\mathrm{X}_{\mathrm{W}}, \mathcal{O})[\![q-1]\!]$. Letting $\mathrm{X}_{\mathrm{W}} = \operatorname{Spf}\mathrm{R}$, this prism induces a flat-local surjection $\operatorname{Spf}\mathrm{A} \to (\mathrm{X}_{\mathrm{W}}[\zeta_{p^{n+1}}])^{\mathbb{\Delta}} \to (\mathrm{X}_{\mathrm{W}}[\zeta_{p^n}])^{\mathbb{\Delta}}$ (the second map using [BL22b] Lemma 6.3; we use $\mathrm{X}_{\mathrm{W}}[\zeta_{p^n}]$ as shorthand for $\operatorname{Spf}\mathrm{R}\widehat{\otimes}\mathrm{W}(k)[\zeta_{p^n}]$). This allows us to study sheaves on $(\mathrm{X}_{\mathrm{W}}[\zeta_{p^n}])^{\mathbb{\Delta}}$ via pullback to this covering. Note that we use ζ_{p^n} here rather than $\zeta_{p^{n+1}}$ for $n \geq 0$ to also treat the unramified case, as otherwise this would be omitted.

To any coherent prismatic crystal $\mathcal{E}$ after evaluation on $(\mathrm{A}, \varphi^n([p]_q))$, using §4.1 in [Liu25] in the unramified case we may produce maps $\nabla_{\mathrm{M},i} : \mathrm{M} \to \mathrm{M}$ satisfying a twisted Leibniz rule

$$\nabla_{\mathrm{M},i}(am) = \gamma_i(a)\nabla_{\mathrm{M},i}(m) + \nabla_i(a)m$$

where γ_i sends $\mathrm{T}_i \mapsto q^{p^{n+1}}\mathrm{T}_i$ and fixes all other T_j, and $\nabla_i : \mathrm{A} \to \mathrm{A}$ is given by $f \mapsto [p]_{q^{p^n}} \frac{\gamma_i(f)-f}{q^{p^{n+1}}\mathrm{T}_i-\mathrm{T}_i}$. These assemble to a full q-connection $\nabla_{\mathrm{M}} = \sum_i \nabla_{\mathrm{M},i}\mathrm{d}\mathrm{T}_i : \mathrm{M} \to \mathrm{M}\otimes \mathrm{q}\Omega^1$, where $\mathrm{q}\Omega^1 = \mathrm{A}\widehat{\otimes}_{\mathrm{W}(k)\langle \mathrm{T}_1^{\pm},\ldots,\mathrm{T}_\ell^{\pm}\rangle[\![q-1]\!]}\Omega^1_{\mathrm{W}(k)\langle \mathrm{T}_1^{\pm},\ldots,\mathrm{T}_\ell^{\pm}\rangle[\![q-1]\!]/\mathrm{W}(k)[\![q-1]\!]}$.

As the prismatic crystal comes from the absolute prismatic site, we can further produce a q-derivation $\partial : \mathrm{M} \to \mathrm{M}$ satisfying

$$\partial(am) = \gamma_{\mathrm{A}}(a)\partial(m) + \partial_{\mathrm{A}}(a)m$$

where γ_{A} sends $q \mapsto q^{p^{n+1}+1}$ and $\partial_{\mathrm{A}} = \frac{\gamma_{\mathrm{A}}-\mathrm{id}}{q^{p^n}-1}$. Note that this differs from the convention in [Liu25] by a unit.

For handling the case of X_{W}, we will need the following lemma. We use $\Phi_{p^k}(q)$ to denote the p^kth cyclotomic polynomial in q. When $k \geq 1$ this agrees with $[p]_{q^{p^{k-1}}}$. Via a gcd calculation one may check that in $\operatorname{Spec} \mathrm{A}[1/p]$ the loci $\mathrm{V}(\Phi_{p^j}(q))$ for $j \geq 1$ are disjoint subsets of $\operatorname{Spec} \mathrm{A}[1/p]$ for an algebra A over $\mathrm{W}(k)[\![q-1]\!]$.

Lemma 3.8. *Let* $\mathrm{A} = \mathrm{W}(k)[\![q-1]\!]$ *and equip it with* ∂_{A} *(as defined above).*

Now suppose $\mathrm{I} \leq \mathrm{A}$ *is a nonzero ideal which is stable under* ∂_{A}*. Then* $\mathrm{V}(\mathrm{I}[1/p]) \subset \operatorname{Spec} \mathrm{A}[1/p]$ *is contained in finitely many of the disjoint loci* $\mathrm{V}(\Phi_{p^j}(q)) \subset \operatorname{Spec} \mathrm{A}[1/p]$ *for* $j > n$*.*

Proof. We are given that I is a nonzero ideal which is stable under $\partial = \partial_{\mathrm{W}(k)[\![q-1]\!]}$, which implies it is stable under $\gamma = \gamma_{\mathrm{W}(k)[\![q-1]\!]}$. Since γ is an automorphism fixing p and must then have $\gamma(\mathrm{I}[1/p]) = \mathrm{I}[1/p]$, we deduce the finitely many minimal primes $\mathfrak{q}_i$ of $\mathrm{I}[1/p]$ have finite γ-orbits. We will now classify such primes.

After inverting p we find $\mathrm{W}(k)[\![q-1]\!][1/p]$ is a PID and we may then assume the minimal primes $\mathfrak{q}_i \leq \mathrm{W}(k)[\![q-1]\!][1/p]$ are generated by $f \in \mathrm{W}(k)[q-1]$ for some irreducible polynomial $f \equiv (q-1)^d \pmod{p}$. As γ sends $q \mapsto q^{p^{n+1}+1}$, by factoring f over $\mathbf{C}_p$ we note its roots in the variable q lie in $\mathfrak{m}+1 \subset \mathcal{O}_{\mathbf{C}_p}$ due to $f(q-1) \equiv (q-1)^d \pmod{p}$. Since $\gamma^j(q) = q^{(p^{n+1}+1)^j}$, to have finite order we require $q \in \mu_{p^\infty}(\mathcal{O}_{\mathbf{C}_p})$ (any other roots of unity fail to lie in $\mathfrak{m}+1$). Thus we allow only $\Phi_{p^j}(q)$ for $j \geq 0$ as possibilities for f when we invert p. It follows that $\mathrm{I}[1/p]$ contains a product of finitely many powers of cyclotomic polynomials $\Phi_{p^j}(q)$.

Finally, we must show we can eliminate the remaining cyclotomic factors $\Phi_{p^j}(q)$ for $0 \leq j \leq n$ from this product lying in $\mathrm{I}[1/p]$. Indeed, since we know $\mathrm{W}(k)[\![q-1]\!][1/p]$ is a PID we may assume $\mathrm{I}[1/p]$ is generated by $\prod_{s\in \mathrm{S}} \Phi_{p^s}(q)$ for some finite set S. If one of these cyclotomic polynomials is $\Phi_{p^j}(q)$ for $0 \leq j \leq n$, we may assume $\mathrm{I}[1/p]$ is generated by $\Phi_{p^j}(q)^N x$ where x is a finite product of other cyclotomic factors. Applying ∂ we get $\gamma(\Phi_{p^j}(q)^N)\partial(x) + \partial(\Phi_{p^j}(q)^N)x \in \mathrm{I}[1/p]$. Using $j \leq n$, one may compute that the first summand is divisible by $\Phi_{p^j}(q)^N$. The second summand is not: we know from the earlier gcd computation that x is coprime to $\Phi_{p^j}(q)^N$ in $\mathrm{W}(k)[\![q-1]\!][1/p]$. A computation shows $\partial(\Phi_{p^j}(q)^N)$ is not divisible by $\Phi_{p^j}(q)^N$. This then gives a contradiction. ☐

The following lemma can then be shown by reducing to the previous one.

Lemma 3.9. *Let* $\mathrm{X_W} \to \operatorname{Spf} \mathrm{W}(k)\langle \mathrm{T}_1^{\pm}, \ldots, \mathrm{T}_\ell^{\pm}\rangle$ *be étale, with* A *the associated* q *de Rham prism. Then if* $\mathcal{E} \in \mathsf{Coh}(\mathrm{X_W}[\zeta_{p^n}]^{\Delta})$*, after evaluating on the prism* $(\mathrm{A}, [p]_{q^{p^n}})$ *the module* $\mathcal{E}(\mathrm{A})[1/p]$ *is a vector bundle away from finitely many of the disjoint loci* $\mathrm{V}(\Phi_{p^j}(q)) \subset \operatorname{Spec} \mathrm{A}[1/p]$ *for* $j > n$*.*

Proof. We first may further reduce to the case that $\mathrm{X_W} = \operatorname{Spf} \mathrm{R}$ is connected finite étale over a basic open of the form $\mathrm{D}(g)$ in $\operatorname{Spf} \mathrm{W}(k)\langle \mathrm{T}_1^{\pm}, \ldots, \mathrm{T}_\ell^{\pm}\rangle$, by shrinking. It suffices to show we obtain a vector bundle on the desired locus after evaluating on the cyclotomic prism $(\mathrm{A}, [p]_{q^{p^n}}) = (\Gamma(\mathrm{X_W}, \mathcal{O})[\![q-1]\!], [p]_{q^{p^n}})$ and inverting p. Evaluating $\mathcal{E}$ on this prism yields a module M equipped with $\nabla_{\mathrm{M},i}$ and ∂. If $\mathrm{I} = \mathrm{Fitt}(\mathrm{M})$ is the first nonzero Fitting ideal it then suffices to show that I is the unit ideal after inverting p and finitely many of $\Phi_{p^j}(q)$ for $j > n$ by using Stacks Project 07ZD. Fitting ideals are stable under base change, so using this property we may claim

$$\partial_{\mathrm{A}}(\mathrm{I}) \subseteq \mathrm{I}, \nabla_{\mathrm{A},i}(\mathrm{I}) \subseteq \mathrm{I}$$

where ∂_{A} and $\nabla_{\mathrm{A},i}$ are the q-derivation and connection on the base prism A. We now show this for ∂_{A}; the proof for $\nabla_{\mathrm{A},i}$ is similar. Let $\mathrm{A}_\varepsilon = \mathrm{A}[\varepsilon]/(\varepsilon^2 - (q^{p^n}-1)\varepsilon)$, which is equipped with a map

$$\sigma = \mathrm{id} + \varepsilon\partial_{\mathrm{A}} : \mathrm{A} \to \mathrm{A}_\varepsilon.$$

The existence of a q-derivation on M yields an isomorphism $\sigma^*\mathrm{M} \simeq \mathrm{M}_\varepsilon$ where on the right the module is the base change $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{A}_\varepsilon$ along the canonical inclusion $\mathrm{A} \to \mathrm{A}_\varepsilon$. Indeed, one sends

$$m \otimes 1 \mapsto m \otimes 1 + \partial_{\mathrm{M}}(m) \otimes \varepsilon$$

and the axioms of a q-derivation make this an A_ε-module isomorphism. Base change of Fitting ideals implies $\mathrm{Fitt}(\mathrm{M}_\varepsilon) = \mathrm{I}\mathrm{A}_\varepsilon$ and $\mathrm{Fitt}(\sigma^*\mathrm{M}) = \sigma(\mathrm{I})\mathrm{A}_\varepsilon$. Equality of these two ideals forces $\partial(\mathrm{I}) \subseteq \mathrm{I}$ by looking at the ε component.

One may also similarly deduce stability of the ideal under $\nabla_{\mathrm{A},i}$. We will show that for $\mathrm{R} = \Gamma(\mathrm{X_W}, \mathcal{O})$ that any ideal

$$\mathrm{I} \leq \mathrm{R}[\![q-1]\!]$$

stable under the operators $\nabla_{\mathrm{A},i}$ in fact contains a nonzero element of $\mathrm{W}(k)[\![q-1]\!]$. We can reduce using the norm map for a finite étale extension to where $\operatorname{Spf} \mathrm{R} = \mathrm{D}(g)$ corresponds to a basic open in the torus $\operatorname{Spf} \mathrm{W}(k)\langle \mathrm{T}_1^{\pm}, \ldots, \mathrm{T}_\ell^{\pm}\rangle[\![q-1]\!]$. Indeed, if $\mathrm{I} \leq \mathrm{R}[\![q-1]\!]$ and $\operatorname{Spf} \mathrm{R} \to \operatorname{Spf} \mathrm{S}$ is finite etale, then using Cayley-Hamilton one sees $\mathrm{Nm}_{\mathrm{R}[\![q-1]\!]/\mathrm{S}[\![q-1]\!]}(f) \in \mathrm{I} \cap \mathrm{S}[\![q-1]\!]$ and further the norm is nonzero if $f \in \mathrm{I}$ is nonzero (we use that R is a domain here so that the multiplication by f map is injective). We also know $\mathrm{I} \cap \mathrm{S}[\![q-1]\!]$ is $\nabla_{\mathrm{A},i}$-stable, so it then suffices to prove the claim for S.

We may reduce to the weaker condition of considering ideals stable under $\gamma_{\mathrm{A},i}$ (as this is implied by $\nabla_{\mathrm{A},i}$-stability). Observe that $\mathrm{R}[\![q-1]\!]$ can equivalently be viewed as a $(p, q-1)$-adic

Tate algebra: this ring identifies with $\mathrm{A} = \mathrm{W}(k)[\![q-1]\!]\langle \mathrm{T}_i^{\pm}, 1/g\rangle_{(p,q-1)}$ where the completion is $(p, q-1)$-adic. Now one can consider the analytic locus with respect to the maximal ideal of $\mathrm{W}(k)[\![q-1]\!]$, defined as the adic space $\mathrm{A}_a := \mathrm{Spa}(\mathrm{A}, \mathrm{A}) \setminus \mathrm{V}(p, q-1)$. For the q-de Rham prism $\mathrm{Spf}\,\mathrm{W}(k)[\![q-1]\!]$ with the $(p, q-1)$-adic topology this construction decomposes as an adic space $\mathrm{Y} = (\mathrm{Spf}\,\mathrm{W}(k)[\![q-1]\!])_a = \mathbb{D}^{\circ}_{W(k)[1/p]} \cup y_{\partial}$, which is quasicompact; the point y_{∂} has residue field $k((q-1))$. The rational neighborhoods $\mathrm{U}_N = \{|p| \leq |q-1|^N\}$ form a basis of opens around y_{∂}. One first p-saturates the ideal I (which preserves stability under $\gamma_{\mathrm{A},i}$) to ensure I/p is nonzero, deducing from $q-1$-torsionfreeness that on A_a the ideal is nonzero in the fiber $\mathrm{A}_{k((q-1))}$ over y_{∂}; one may show I is forced to become the unit ideal in this fiber $(\mathrm{A}_a)_{y_{\partial}}$, meaning that $\mathrm{V}(\mathrm{I}) \subset \mathrm{A}_a$ does not intersect some U_N. Indeed, if the ideal is proper, pick a nonzero point in the vanishing locus and choose a small polydisk centered around it. By studying the action of γ_i, since $|q-1| < 1$ applying sufficiently high powers of γ_i we force the polydisk to contain infinitely many zeroes in each coordinate direction, forcing vanishing via the rigid analytic identity theorem.

Take some nonzero $f \in \mathrm{I}$ and write $f = \sum_{n\geq 0} f_n(q-1)^n$. Let n_0 be the smallest integer such that $f_{n_0} \neq 0$, and choose a closed point x away from the zero locus of f_{n_0} in $\mathrm{Spa}(\mathrm{R}[1/p], \mathrm{R})$ induced by a map $\mathrm{R} \to \mathcal{O}_L$. This induces an evaluation map $\mathrm{R}[\![q-1]\!] \to \mathcal{O}_L[\![q-1]\!]$; taking the norm of this evaluation map applied to f produces $c \in \mathrm{W}(k)[\![q-1]\!]$ with the property that $c(y) \neq 0$ implies $\mathrm{I}_y \neq 0$ for the fiber over $y \in \mathrm{Y}$. On points y away from U_N we may again analyze fibers, using the same method to deduce that I_y is the unit ideal unless $q(y)$ is a root of unity. Away from U_N there are only finitely many such points as we have $|(q-1)(y)| < |p|^{1/N}$, constraining the support of A/I to the zero locus of $c(q-1)\Pi_N(q)$ where Π_N is some product of cyclotomic factors. From this we deduce the desired claim.

Once we know $\mathrm{I} \cap \mathrm{W}(k)[\![q-1]\!]$ is nonzero, we observe the intersection remains ∂_{A}-stable. So Lemma 3.8 applies, and the desired claim follows. ☐

For an ordinary connection (as opposed to a q-connection) we have the following result using [And01] Corollary 2.5.2.2.

Corollary 3.10. *Let* $\mathrm{X}/\,\mathrm{Spf}\,\mathrm{W}(k)$ *be smooth and quasicompact. If* $\mathrm{M} \in \mathsf{Coh}(\mathrm{X})$ *admits a connection, then* $\mathrm{M}[1/p]$ *is a vector bundle.*

Proof. As X is smooth and quasicompact, there is a finite covering by formal smooth affines and it suffices to prove the claim for these. In this case, taking the adic generic fiber induces a coherent sheaf with connection on the affinoid generic fiber, which is then forced to be a vector bundle by applying [And01] Corollary 2.5.2.2. ☐

This torsion control also implies Proposition 3.7 in the cyclotomic case. We may test the claim of Proposition 3.7 Zariski locally in X. We may then assume that $\mathrm{X} = \mathrm{X}_{\mathrm{W}[\zeta_{p^n}]}$ as in Lemma 3.9. Let I denote the first nonzero Fitting ideal of the module M we get after evaluation on the corresponding cyclotomic prism $(\mathrm{A}, [p]_{q^{p^n}})$. Our goal is to show $\mathrm{I}[1/p] = \mathrm{A}[1/p]$ using the prismatic F-crystal structure, so then applying Stacks Project 07ZD the desired claim follows (using that $(\mathrm{X}/\mathrm{A})^{\mathbb{\Delta}} \to \mathrm{X}^{\mathbb{\Delta}}$ is a flat-local surjection). We may use $\mathrm{I}[1/[p]_{q^{p^n}}] \simeq \varphi(\mathrm{I})[1/[p]_{q^{p^n}}]$, which follows from stability of Fitting ideals under base change. Recall that via a gcd calculation one may check in $\operatorname{Spec} \mathrm{A}[1/p]$ that the loci $\mathrm{V}(\Phi_{p^j}(q))$ for $j \geq 1$ are disjoint subsets of $\operatorname{Spec} \mathrm{A}[1/p]$. Now I is not stable under the faithfully flat map φ, but $\mathrm{I}[1/\Phi_{p^{n+1}}(q)] = \varphi(\mathrm{I})[1/\Phi_{p^{n+1}}(q)]$ implies that $\mathrm{V}(\varphi(\mathrm{I}))$ agrees with $\mathrm{V}(\mathrm{I})$ restricted to $\operatorname{Spec} \mathrm{A}[1/p, 1/\Phi_{p^{n+1}}(q)]$. We know

$$\mathrm{V}(\mathrm{I}[1/p]) \subset \bigcup_{j \in \mathrm{S}} \mathrm{V}(\Phi_{p^j}(q))$$

as loci in $\operatorname{Spec} \mathrm{A}[1/p]$ for some finite subset $\mathrm{S} \subset \mathbf{Z}_{>n}$ by Lemma 3.9; we choose the finite set $\mathrm{S} \subset \mathbf{Z}_{>n}$ to be minimal. Since φ is faithfully flat and the preimage of $\mathrm{V}(\Phi_{p^j}(q))$ is $\mathrm{V}(\Phi_{p^{j+1}}(q))$ in $\operatorname{Spec} \mathrm{A}[1/p]$, we learn $\mathrm{S} \setminus \{n+1\} = (\mathrm{S}+1) \setminus \{n+1\}$, forcing $\mathrm{S} = \emptyset$. That is, $\mathrm{V}(\mathrm{I}[1/p])$ is empty in $\operatorname{Spec} \mathrm{A}[1/p]$ as desired.

3.1.2. *Proving Theorem 3.5.* Next, we apply Proposition 3.7 to understand the kernel of the étale realization to work towards Theorem 3.5.

Proposition 3.11. *Assume* X *is smooth and quasicompact over* $\operatorname{Spf} \mathcal{O}_K$ *and let* $\mathcal{E}$ *be a coherent prismatic* F*-crystal. If* $\mathrm{T}_{\text{ét}}(\mathcal{E})$ *is* p*-power torsion, then so is* $\mathcal{E}$*.*

Proof. We may work locally, so we can assume that X is smooth affine so that $\mathrm{X}^{\mathbb{\Delta}}$ has a Breuil-Kisin prism (A, I) where $\rho : \operatorname{Spf} \mathrm{A} \to \mathrm{X}^{\mathbb{\Delta}}$ is a flat-local surjection. Using Proposition 3.7 and letting $\mathrm{A}_{\mathrm{perf}}$ denote the perfection of A, the pullback of $\mathcal{E}|_{\mathrm{X}^{\mathbb{\Delta}}}$ to $\operatorname{Spf} \mathrm{A}_{\mathrm{perf}}$ must then be a vector bundle over $\mathrm{A}_{\mathrm{perf}}[1/p] = \mathrm{W}(\mathrm{R}^\flat_{\mathrm{perf}})[1/p]$. For a perfectoid ring, there is an equivalence

$$\mathrm{T}_{\text{ét}}[1/p] : \mathsf{Vect}(\mathrm{W}(\mathrm{R}^\flat_{\mathrm{perf}})[1/\mathrm{I}]^\wedge_p[1/p])^{\varphi=1} \simeq \mathsf{Loc}_{\mathbf{Z}_p}(\mathrm{R}_{\mathrm{perf},\eta})[1/p].$$

Since $\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]$ is assumed to be zero, it follows $\mathcal{E}(\mathrm{W}(\mathrm{R}^\flat_{\mathrm{perf}}), \mathrm{I})[1/\mathrm{I}]^\wedge_p[1/p]$ is zero. Hence we conclude $\mathcal{E}(\mathrm{W}(\mathrm{R}^\flat_{\mathrm{perf}}), \mathrm{I})[1/\mathrm{I}]^\wedge_p$ is p-power torsion. It follows by p-completely flat descent that the F-crystal $\mathcal{E}(\mathrm{A}, \mathrm{I})[1/\mathrm{I}]^\wedge$ is p-power torsion by replacing $\mathcal{E}$ with its p-torsionfree quotient $\mathcal{E}^{p-\mathrm{tf}}$ and using descent to conclude this is zero (the Frobenius on A is faithfully flat, so $\operatorname{Spf} \mathrm{A}_{\mathrm{perf}} \to \operatorname{Spf} \mathrm{A}$ is a flat cover). Finally, knowing that $\mathcal{E}(\mathrm{A}, \mathrm{I})[1/p]$ is a vector bundle by Proposition 3.7 this forces $\mathcal{E}(\mathrm{A}, \mathrm{I})[1/p] = 0$ (as it would be a projective module which becomes zero after an injective base change), after which the flat-local surjection ρ proves the claim. ☐

We can now understand the kernel of $\mathrm{T}_{\text{ét}}$ using the argument in Remark 6.3.5 of [Bha22].

Proof of Theorem 3.5. It is easy to check that (p, v_1)-power torsion objects are killed by $\mathrm{T}_{\text{ét}}$. Let $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$ be killed by $\mathrm{T}_{\text{ét}}$. First, we note that $\mathcal{E}/p\mathcal{E}$ is $v_{1,\mathrm{X}}$-torsion. As X is smooth over $\mathcal{O}_K$, picking a framing reduces us to the case where $\mathrm{X} = \operatorname{Spf} \mathrm{R}$ is étale over $\operatorname{Spf} \mathcal{O}_K\langle \mathrm{X}_1^\pm, \dots, \mathrm{X}_d^\pm \rangle$ and thus admits a quasisyntomic cover by a perfectoid $\operatorname{Spf} \tilde{\mathrm{R}}$. We may check the $v_{1,\mathrm{X}}$-torsion claim locally, and on a perfectoid this is obvious since étale realization simply inverts $v_{1,\tilde{\mathrm{R}}}$ for p-torsion F-gauges.

Next, we must show that $\mathcal{E}$ is p-power torsion. We know that $\mathcal{E}|_{\mathrm{X}^{\mathbb{\Delta}}}$ is p-power torsion using Proposition 3.11, as the étale realization is zero and compatible with restriction to $\mathrm{X}^{\mathbb{\Delta}}$, and then we can deduce $\mathcal{E}$ is p-power torsion analogously to Remark 6.3.5 of [Bha22], except we use a cover of $\mathrm{X}^{\mathrm{Syn}}$.

By quasicompactness, we may test the p-power torsion claim Zariski locally for $\mathrm{X} = \operatorname{Spf}\mathrm{R}$ using a Breuil-Kisin prism $(\mathrm{A}, (d))$. In what follows we will denote the prism in either case $(\mathrm{A}, (d))$, as the argument is the same until we require a connection. Consider the associated cover

$$\rho_{\mathrm{A}} : \mathrm{Rees}_{\mathrm{I}^{\bullet}}(\mathrm{A}) \to \mathrm{X}^{\mathrm{Syn}}$$

and identify the Rees stack with $(\operatorname{Spf}\mathrm{A}\langle u, t\rangle/(ut - d))/\mathbf{G}_m$ where $\deg(t) = 1, \deg(u) = -1$. We simplify the argument by replacing $\mathcal{E}$ by its p-torsionfree quotient, and we now want to show it pulls back to 0 on this cover (in which case we are done by flat descent). We then know the restriction to $\operatorname{Spf}\mathrm{A}\langle t^{\pm}\rangle/\mathbf{G}_m$ is zero as this is the corresponding cover of $\mathrm{X}^{\mathbb{\Delta}}$ for pullback along j_{dR}^*, so we deduce that $\mathrm{M} = \rho_{\mathrm{A}}^*\mathcal{E}$ is t-power torsion (here we use the grading).

Now filtering by powers of t, we can assume $t\mathrm{M} = 0$ or that M is a sheaf on

$$\operatorname{Spf}(\mathrm{A}\langle u, t\rangle/(ut - d))/\mathbf{G}_m|_{t=0} \simeq [\operatorname{Spf}\mathrm{R}\langle u\rangle/\mathbf{G}_m].$$

The sheaf M vanishes after pullback to $[\operatorname{Spf}\mathrm{R}\langle u^{\pm}\rangle/\mathbf{G}_m]$ as the pullback along j_{HT}^* also vanishes by virtue of $\mathcal{E}$ being an F-gauge: both $j_{\mathrm{dR}}^*, j_{\mathrm{HT}}^*$ identify with $j_{\mathbb{\Delta}}^*$ which we know vanishes. We obtain a finitely generated graded module M over $\mathrm{R}\langle u\rangle$ such that $\mathrm{M}[1/u]_p^{\wedge} = 0$. We will show a variant of a graded connection on $\mathrm{M}^{(1)} \in \mathsf{Coh}((\mathrm{Rees}_{\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}}\mathrm{A}^{(1)})_{t=0})$ arises naturally from $\mathcal{E}$ on $\mathsf{Coh}(\mathrm{X}^{\mathrm{Nyg}})$. We will use this to deduce $\mathrm{M}[1/p] = 0$, which since we replaced $\mathcal{E}$ by its p-torsionfree quotient forces $\rho_{\mathrm{A}}^*\mathcal{E} = 0$ as desired.

Write $(\mathrm{A}, (d)) = (\mathrm{R}_0[\![u_0]\!], \mathrm{E}(u_0))$ where $\mathrm{E}(u_0)$ is an Eisenstein polynomial for a uniformizer π of $\mathcal{O}_K$. We may adapt the Nygaard-filtered variant of ∇_{M} from Definition 4.10 to the module $\mathrm{M}^{(1)}$. We may use the same construction, just replacing $\mathrm{W}(k)$ with R_0: we have $\Gamma_0 = \mathrm{R}_0[\![u_0]\!]$ and $\Gamma_1 = \mathrm{R}_0[\![u_0, u_1]\!]\left\{\frac{u_0 - u_1}{\mathrm{E}(u_0)}\right\}^{\wedge}$ where we take the $(p, \mathrm{E}(u_0))$-adic completion and also F-complete. The map Θ can be constructed completely analogously, just now using an R-linear basis. The connection is similarly Nygaard filtered.[2]

Passing to the Nygaard associated graded, we obtain an induced map $\overline{\nabla}_{\mathrm{M}}$ on

$$\mathrm{Rees}_{\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}}(\mathrm{A}^{(1)})_{t=0} \simeq \mathrm{A}/\mathrm{E}(u_0)\langle u\rangle/\mathbf{G}_m.$$

For $\mathrm{M}^{(1)}$ coming from the structure sheaf on $\mathcal{O}_K^{\mathrm{Nyg}}$, we may express $\nabla = \Theta \circ (\eta_R - \eta_L)$ in the notation of [AKN24], and then on the twist $\mathrm{W}(k)[\![u_0]\!]^{(1)}$ the Nygaard filtration has $\mathrm{Fil}^i_{\mathrm{Nyg}}$ generated by $u^i := 1 \otimes \mathrm{E}(u_0)^i$; following these elements through the definition of ∇, one expands $(\eta_R - \eta_L)(1 \otimes \mathrm{E}(u_0)^i) = 1 \otimes (\mathrm{E}(u_1)^i - \mathrm{E}(u_0)^i)$ and applies Θ to deduce $\overline{\nabla}$ satisfies $\mathrm{gr}^{i-1}_{\mathrm{Nyg}}\overline{\nabla}(u^i) = i\mathrm{E}'(\pi)u^{i-1}$. In particular one uses a similar argument as in Lemma 4.51(ii) to obtain divided powers of $g_0 := u_1 - u_0$ and rewrite powers g_0^n for $n \geq p$ in the g_u

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[2] It is worth noting in the general case ∇_{M} doesn't capture the full data of a prismatic crystal: using the analogous descent complex as in the discussion preceding Lemma 4.9, this is specifying instead the data of a coherent sheaf on $(\mathrm{X}/\mathrm{R}_0)^{\mathbb{\Delta}}$, the prismatic stack relative to the δ-ring R_0 as in [AKN23]. However we only need ∇_{M} in the argument.

basis that Θ is defined in; the identity for simplifying g_u^p in the g_u basis actually becomes $g_u^p = -pg_{u+1}$ in the Nygaard associated graded. Using these divided powers we may Taylor expand $\mathrm{E}(u_1)^i$, and then applying Θ the only term contributing is $1 \otimes i\mathrm{E}'(u_0)\mathrm{E}(u_0)^{i-1}g_0$, which is sent by Θ to the desired output $i\mathrm{E}'(\pi)u^{i-1}$ on the associated graded.

We may twist $\mathrm{M}^{(1)}$ so that $\mathrm{Fil}^0_{\mathrm{Nyg}}\mathrm{M}^{(1)} = \mathrm{M}^{(1)}$ for convenience. For general $\mathrm{M}^{(1)}$, since on the Nygaard associated graded we have $g_u^p = -pg_{u+1}$ we deduce the Hopf algebroid $(\mathrm{gr}^*_{\mathrm{Nyg}}\Gamma_0^{(1)}, \mathrm{gr}^*_{\mathrm{Nyg}}\Gamma_1^{(1)})$ (with Γ_0, Γ_1 as in [AKN24] Section 4.6 for $\mathrm{R} = \mathcal{O}_K$; see §4.2 for more details) has $\mathrm{gr}^*_{\mathrm{Nyg}}\Gamma_1^{(1)}$ given as a free divided power algebra in g_0 over $\mathrm{gr}^*_{\mathrm{Nyg}}\Gamma_0^{(1)}$ with g_0 primitive, and moreover Θ is similar to Example 4.49 in [AKN24] since now $\Theta(g_0^{[n]}) = 0$ unless $n = 1$ (where we get 1). This situation suffices to deduce a Leibniz rule for a general finitely presented comodule $\mathrm{M}^{(1)}$. Define $\overline{\nabla}_{\mathrm{M}} := (1 \otimes \Theta) \circ \rho$. Writing $\rho(m) = \sum_{n \geq 0} m_n \otimes g_0^{[n]}$, we have $m_0 = m$ by counitality, and we write $\eta_R(a) = \sum_{i \geq 0} \overline{\nabla}^{[i]}(a) g_0^{[i]}$ for $a \in \mathrm{gr}^*_{\mathrm{Nyg}}\Gamma_0^{(1)}$. Here $\overline{\nabla}^{[i]}(a)$ are by definition the coefficients of $g_0^{[i]}$, agreeing with $\overline{\nabla}$ when $i = 1$ and giving the identity when $i = 0$. Hence

$$\rho(ma) = \rho(m)\eta_R(a) = \sum_{i,j \geq 0} m_j \overline{\nabla}^{[i]}(a) \otimes g_0^{[j]} g_0^{[i]}.$$

Using $g_0^{[i]} g_0^{[j]} = \binom{i+j}{i} g_0^{[i+j]}$, the only terms contributing to the $g_0^{[1]}$-coefficient are $(i,j) = (1,0)$ and $(0,1)$. Moreover, from the expansion of $\rho(m)$ and the definition $\overline{\nabla}_{\mathrm{M}} = (1 \otimes \Theta) \circ \rho$ we have $m_1 = \overline{\nabla}_{\mathrm{M}}(m)$. Thus this coefficient is $m\overline{\nabla}(a) + m_1 a = m\overline{\nabla}(a) + \overline{\nabla}_{\mathrm{M}}(m)a$. Since $\Theta(g_0^{[n]}) = 0$ for $n \neq 1$ and $\Theta(g_0) = 1$, this coefficient is exactly $\overline{\nabla}_{\mathrm{M}}(ma)$, giving the Leibniz rule, or equivalently $\overline{\nabla}_{\mathrm{M}}(am) = \overline{\nabla}(a)m + a\overline{\nabla}_{\mathrm{M}}(m)$.

Since $\mathrm{M}^{(1)}[1/u]^\wedge = 0$ and since it is a *graded* module over $\mathrm{A} \otimes_{\mathrm{A},\varphi} \mathrm{R}\langle u \rangle$ we may conclude it must be u-torsion (again using the grading). Now consider $\mathrm{M}^{(1)}[1/p]$. Take the minimal $i > 0$ such that $u^i m = 0$ for all m, and then applying $\overline{\nabla}_{\mathrm{M}}$ to both sides contradicts minimality by showing $0 = i\mathrm{E}'(\pi)u^{i-1}m$; the factor in front is now a unit since we inverted p (the other factor coming from the Leibniz rule vanishes, being divisible by u^i). Thus after we invert p the module is forced to be zero and thus $\mathrm{M}[1/p] = 0$ as desired. ☐

3.2. **Reflexive F-gauges.**

We can define reflexive prismatic F-crystals in a similar way to reflexive F-gauges.

Definition 3.12. *A coherent prismatic F-crystal M on a smooth p-adic formal scheme over $\mathcal{O}_K$ is reflexive if the canonical map $\mathrm{M} \to \mathrm{M}^{\vee\vee}$ is an equivalence. We let $\mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ denote the category of reflexive prismatic F-crystals.*

Here, $(-)^\vee$ means we take the $\mathcal{O}_{\Delta}$-linear dual and endow it with the structure of a coherent prismatic F-crystal.

There is something to check in this definition, namely that the $\mathcal{O}_{\Delta}$-linear dual still has the structure of a coherent prismatic F-crystal for $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ smooth. For a prism (A, I) and

$\mathrm{M} \in \mathsf{Perf}^{\heartsuit}(\mathrm{A})$ such that $\varphi^*\mathrm{M}[1/\mathrm{I}] \simeq \mathrm{M}[1/\mathrm{I}]$ we claim that

$$\varphi^*\mathrm{Hom}(\mathrm{M}, \mathrm{A})[1/\mathrm{I}] \simeq \mathrm{Hom}(\mathrm{M}, \mathrm{A})[1/\mathrm{I}]$$

if the Frobenius φ on A is flat.

Observe $\mathrm{Hom}(\mathrm{M}, \mathrm{A})[1/\mathrm{I}] = \mathrm{Hom}(\mathrm{M}[1/\mathrm{I}], \mathrm{A}[1/\mathrm{I}])$. Then

$$\varphi^*\mathrm{Hom}(\mathrm{M}[1/\mathrm{I}], \mathrm{A}[1/\mathrm{I}]) \simeq \mathrm{Hom}(\varphi^*\mathrm{M}[1/\mathrm{I}], \mathrm{A}[1/\mathrm{I}]) \simeq \mathrm{Hom}(\mathrm{M}[1/\mathrm{I}], \mathrm{A}[1/\mathrm{I}])$$

where we used flatness and $\varphi^*\mathrm{A}[1/\mathrm{I}] \simeq \mathrm{A}[1/\mathrm{I}]$ in the first isomorphism and the assumption on M that $\varphi^*\mathrm{M}[1/\mathrm{I}] \simeq \mathrm{M}[1/\mathrm{I}]$ in the second. For X smooth over $\mathrm{Spf}\, \mathcal{O}_K$, we can describe $\mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ by descent using Breuil-Kisin prisms. The Frobenius on such a prism is faithfully flat by Corollary 3.6 in [GL23], and the same will hold for prisms in the Čech nerve. Thus, we have a good notion of duals.

For the following lemma, we regard a reflexive module as a module M in $\mathsf{Perf}^{\heartsuit}(\mathrm{A})$ equipped with an isomorphism $\mathrm{M} \simeq \mathrm{M}^{\vee\vee}$, and denote this category by $\mathsf{Refl}(\mathrm{A})$. For a prism (A, I), we write

$$\mathsf{Refl}^{\varphi}(\mathrm{A}) = \{(\mathrm{M}, f) : \mathrm{M} \in \mathsf{Refl}(\mathrm{A}), f : \varphi_{\mathrm{A}}^*\mathrm{M}[1/\mathrm{I}] \simeq \mathrm{M}[1/\mathrm{I}]\}.$$

Lemma 3.13. *The category* $\mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ *satisfies* p*-complete étale descent in smooth* p*-adic formal schemes over* $\mathcal{O}_K$*.*

If X *is smooth affine, then given any Breuil-Kisin prism* (A, I) *for* X *we have*

$$\mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \simeq \lim_{n \in \Delta} \mathsf{Refl}^{\varphi}(\mathrm{A}^{(n)})$$

where $\mathrm{A}^{(n)}$ *is the* n*th term in the Čech nerve of* (A, I)*.*

Proof. We argue these claims for $\mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$, as the assertions for reflexive objects in this category follow formally from the descent equivalences being compatible with duals.

For étale descent, let $f : \tilde{\mathrm{X}} \to \mathrm{X}$ be an étale covering. Then $\tilde{\mathrm{X}}$ remains smooth, so we have a well-defined t-structure on $\mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$. As $\mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ satisfies quasisyntomic descent (Proposition 2.14 in [BS23]), it also satisfies descent for this covering. Now it suffices to check that the pullback map $f^* : \mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \to \mathsf{Perf}^{\varphi}(\tilde{\mathrm{X}}_{\Delta}, \mathcal{O}_{\Delta})$ is t-exact, which can be seen from Proposition 3.11 in [GL23].

In the situation where X is smooth affine with a Breuil-Kisin prism (A, I), we have

$$\mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \simeq \lim_{n \in \Delta} \mathsf{Perf}^{\varphi}(\mathrm{A}^{(n)}).$$

The claim for $\mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ follows because the maps $\mathrm{A} \to \mathrm{A}^{(n)}$ are flat (rather than just (p, I)-completely flat) as the source is Noetherian and both source and target are derived (p, I)-complete. ☐

We now show that when $\mathrm{X}/\, \mathrm{Spf}\, \mathcal{O}_K$ is smooth that reflexive prismatic F-crystals coincide with analytic prismatic F-crystals.

Proposition 3.14. *Let* X *be étale over* $\mathrm{Spf}\,\mathcal{O}_K\langle \mathrm{X}_1^{\pm}, \dots, \mathrm{X}_n^{\pm}\rangle$*, and let* (A, I) *be a Breuil-Kisin prism associated to* X*. If* $\mathcal{E} \in \mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$*, then*

$$\mathcal{E}(\mathrm{A}, \mathrm{I})|_{\mathrm{Spec}\,\mathrm{A}\setminus \mathrm{V}(p,\mathrm{I})}$$

is a vector bundle.

Proof. We observe that $\mathrm{T}_{\text{ét}}(\mathcal{E})$ is a reflexive object in $\mathrm{D}^{(b)}_{\text{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)^{\heartsuit}$ due to compatibility with duals and t-exactness of $\mathrm{T}_{\text{ét}}$ (see [GL23] Lemma 3.16; we use the underived dual). This implies it is in the subcategory $\mathsf{Loc}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ as the condition $\mathbb{L} \simeq \mathbb{L}^{\vee\vee}$ implies $\mathbb{L} \in \mathrm{D}^{(b)}_{\text{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)^{\heartsuit}$ has no p-torsion.

By [BS23] Corollary 3.8, there is an equivalence

$$\mathsf{Vect}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathrm{I}_{\Delta}]^{\wedge}_p)^{\varphi=1} \simeq \mathsf{Loc}_{\mathbf{Z}_p}(\mathrm{X}_\eta).$$

The étale realization will factor through this equivalence, so we deduce $\mathcal{E}[1/\mathrm{I}_{\Delta}]^{\wedge}$ is a vector bundle. Thus if $\mathrm{M} = \mathcal{E}(\mathrm{A}, \mathrm{I})$ then $\mathrm{M}[1/\mathrm{I}]^{\wedge}_p$ is a vector bundle.

As Proposition 3.7 applies to any prismatic F-crystal, we deduce from it that $\mathrm{M}[1/p]$ is a vector bundle. By Beauville-Laszlo gluing[3], we have

$$\mathsf{Vect}(\mathrm{A}[1/\mathrm{I}]) \simeq \mathsf{Vect}(\mathrm{A}[1/\mathrm{I}, 1/p]) \times_{\mathsf{Vect}(\mathrm{A}[1/\mathrm{I}]^{\wedge}_p[1/p])} \mathsf{Vect}(\mathrm{A}[1/\mathrm{I}]^{\wedge}_p)$$

so we deduce that $\mathrm{M}[1/\mathrm{I}]$ is a vector bundle. It then follows that $\mathrm{M}|_{\mathrm{Spec}\,\mathrm{A}\setminus\mathrm{V}(p,\mathrm{I})}$ is a vector bundle. ☐

We can now use the étale descent of $\mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ to finish.

Theorem 3.15. *Let* $\mathcal{E}$ *be a coherent prismatic* F*-crystal on some smooth* $\mathrm{X}/\,\mathrm{Spf}\,\mathcal{O}_K$*. Then* $\mathcal{E}$ *is reflexive if and only if* $\mathcal{E}$ *is an analytic prismatic* F*-crystal.*

Proof. First, we claim there is a fully faithful inclusion functor

$$\mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \to \mathsf{Refl}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$$

where the target is the category of reflexive prismatic F-crystals. Full faithfulness is automatic, so we only need to show analytic prismatic F-crystals are reflexive. By Zariski descent of both categories (see Lemma 3.13), we can assume X is étale over $\mathcal{O}_K\langle \mathrm{X}_1^{\pm}, \dots, \mathrm{X}_n^{\pm}\rangle$ by choosing a framing. We can choose a Breuil-Kisin prism (A, I) so $\mathrm{Spf}\,\mathrm{A}/\mathrm{I} \simeq \mathrm{X}$. Noting that we can test reflexivity after evaluation on this prism by the second claim in Lemma 3.13, we use that any $*$-extension of a vector bundle on $\mathrm{Spec}\,\mathrm{A}\setminus\mathrm{V}(p,\mathrm{I})$ is a reflexive module over A as A is Noetherian regular. This is shown in Stacks Project 0EBJ.

Next, it suffices to show that for every reflexive prismatic F-crystal $\mathcal{E}$ that

$$\mathrm{M} := \mathcal{E}(\mathrm{A}, \mathrm{I}) \in \mathsf{Vect}^{\varphi}(\mathrm{Spec}\,\mathrm{A} \setminus \mathrm{V}(p, \mathrm{I})).$$

---

[3] In the Noetherian situation we are working in, this amounts to faithfully flat descent via the map $\mathrm{A}[1/\mathrm{I}] \to \mathrm{A}[1/\mathrm{I}]^{\wedge}_p \times \mathrm{A}[1/\mathrm{I}, 1/p]$, as completion is flat.

Indeed, by the descent results of Lemma 3.13 as well as the analogous results for analytic prismatic F-crystals shown in [GR24], this allows us to verify $\mathcal{E}$ is an analytic prismatic F-crystal as it will automatically lie in $\mathsf{Vect}^{\varphi}(\operatorname{Spec} \mathrm{A}^{(n)} \setminus \mathrm{V}(p, \mathrm{I}))$ for each transversal prism in the Čech nerve of (A, I). As X is assumed étale over $\mathcal{O}_K\langle \mathrm{X}_1^{\pm}, \dots, \mathrm{X}_n^{\pm}\rangle$, we know by Proposition 3.14 that $\mathrm{M} \in \mathsf{Vect}^{\varphi}(\operatorname{Spec} \mathrm{A} \setminus \mathrm{V}(p, \mathrm{I}))$, as $\varphi^*\mathrm{M}[1/\mathrm{I}] \simeq \mathrm{M}[1/\mathrm{I}]$ and $\mathrm{M}|_{\operatorname{Spec} \mathrm{A} \setminus \mathrm{V}(p,\mathrm{I})}$ is a vector bundle. □

We can now characterize reflexive sheaves on $\mathrm{X}^{\mathrm{Syn}}$ as being precisely those lifted from analytic prismatic F-crystals via Π_{X}.

Lemma 3.16. *The functor* Π_{X}

$$\mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \to \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$$

has essential image in $\mathrm{Refl}(\mathrm{X}^{\mathrm{Syn}})$.

Proof. To check coherence, we may reduce to the affine case (the functor in Theorem 3.3 is compatible with étale pullbacks). Then checking coherence amounts to verifying that the pullback to the stack $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A} \to \mathrm{X}^{\mathrm{Nyg}} \to \mathrm{X}^{\mathrm{Syn}}$ is coherent for a Breuil-Kisin prism (A, I)[4]. Let $(\mathrm{A}_{\mathrm{perf}}, \mathrm{I}_{\mathrm{perf}})$ denote the perfection of this prism, so $\mathrm{A}_{\mathrm{perf}} = (\operatorname{colim}_{\varphi} \mathrm{A})^{\wedge}_{(p,\mathrm{I})}$. Since the Frobenius on A is faithfully flat by Corollary 3.6 in [GL23], $\mathrm{A} \to \operatorname{colim}_{\varphi} \mathrm{A}$ is faithfully flat and $\mathrm{A} \to \mathrm{A}_{\mathrm{perf}}$ is derived (p, I)-completely faithfully flat.

We may argue the map of Rees stacks in this case is actually faithfully flat for their natural algebraizations since in

$$\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}_{\mathrm{perf}} \to \mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A} \to \mathrm{X}^{\mathrm{Syn}},$$

the (algebraization of) the target of the first map is a Noetherian algebraic stack. Indeed we may test this on the Rees algebra covers, where the claim we need to argue is that a (p, I)-completely faithfully flat map $\mathrm{R} \to \mathrm{R}'$ of derived (p, I)-complete rings R and R' is further faithfully flat if R is Noetherian. The flatness claim follows from Lemma 5.15 in [Bha20], since the source as a ring map is Noetherian and derived (p, I)-complete. It is further faithfully flat since maximal ideals of R contain (p, I) (since (p, I) lies in the Jacobson radical), and then since we know surjectivity of $\operatorname{Spec} \mathrm{A}_{\mathrm{perf}}/(p, \mathrm{I}) \to \operatorname{Spec} \mathrm{A}/(p, \mathrm{I})$ this suffices.

The map of algebraizations of Rees stacks $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}_{\mathrm{perf}} \to \mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}$ is faithfully flat. It follows that since the pullback to $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}_{\mathrm{perf}}$ is discrete (i.e. in the heart for the t-structure on $\mathrm{D}_{\mathrm{qc}}(\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}_{\mathrm{perf}})$) that the pullback to $\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}$ lies in the heart and hence is coherent.

Having verified Π_{X} sends coherent prismatic F-crystals to coherent F-gauges, we may also easily check Π_{X} is compatible with duals. Since any object $\mathcal{E} \in \mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})$ has $\mathcal{E} \simeq \mathcal{E}^{\vee\vee}$ by Theorem 3.15 the same follows for $\Pi_{\mathrm{X}}(\mathcal{E})$. □

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[4] The reason why it is unclear that the notion of coherence [GL23] uses and the one we use are equivalent is that their notion imposes that the pullback of $\mathcal{E}$ lies in the heart of $\mathsf{Perf}(\operatorname{Spf} \mathrm{S}^{\mathrm{Syn}})$ for every quasiregular semiperfectoid ring $\operatorname{Spf} \mathrm{S} \to \mathrm{X}$, and our notion of coherence implies this only for those quasiregular semiperfectoid rings related to the perfection of a Breuil-Kisin prism.

Theorem 3.17. *The functor* Π_{X} *induces an equivalence of categories*

$$\mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \simeq \mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}).$$

Proof. What remains is to show essential surjectivity. We need to show any $\mathcal{E} \in \mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}})$ is isomorphic to Π_{X} applied to an analytic prismatic F-crystal. It suffices to show that the unit of the adjunction,

$$\eta : \mathcal{E} \to \Pi_{\mathrm{X}}(\mathcal{E}|_{\mathrm{X}^{\Delta}}),$$

is an equivalence since $\Pi_{\mathrm{X}}(\mathcal{E}|_{\mathrm{X}^{\Delta}})$ is reflexive. Indeed, $\mathrm{M} = \mathcal{E}|_{\mathrm{X}^{\Delta}}$ must be a reflexive prismatic F-crystal and hence an analytic prismatic F-crystal by Theorem 3.15.

Let us consider the case where X is smooth affine and integral. We can deduce the general affine case from this via Zariski descent using Stacks Project 0357, and further Zariski descent would show the general case where $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth. We may then write $\mathrm{X} = \operatorname{Spf}\mathrm{A}/\mathrm{I}$ for a Breuil-Kisin prism (A,I) where A is integral. To check η is an equivalence, it suffices to check the induced map on the flat-local surjection $\mathrm{Rees}_{\mathrm{I}^\bullet \mathrm{A}}$ is an equivalence. Let U be the complement of $\mathrm{V}(t,p)$ viewed as a substack of the algebraization $(\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A})^{\mathrm{alg}} := (\operatorname{Spec}\mathrm{A}\langle u,t\rangle/(ut-d))/\mathbf{G}_m$ when $\mathrm{I} = (d)$. As we are dealing with coherent sheaves on Rees stacks and A is an adic Noetherian ring, $\mathsf{Coh}(\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A}) \simeq \mathsf{Coh}((\mathrm{Rees}_{\mathrm{I}^\bullet}\mathrm{A})^{\mathrm{alg}})$.

The map η induces an equivalence when restricted to the $t \neq 0$ substack which covers X^{Δ} (as the underlying prismatic F-crystal of $\Pi_{\mathrm{X}}(\mathrm{M})$ is M). The map $\eta : \mathcal{E} \to \Pi_{\mathrm{X}}(\mathcal{E}|_{\mathrm{X}^{\Delta}})$ induces the identity on étale realization, so the exact sequence

$$0 \to \ker(\eta) \to \mathcal{E} \to \Pi_{\mathrm{X}}(\mathcal{E}|_{\mathrm{X}^{\Delta}}) \to \operatorname{coker}\eta \to 0$$

is sent to $0 \to \mathrm{T}_{\text{ét}}(\mathcal{E}) \to \mathrm{T}_{\text{ét}}(\mathcal{E}) \to 0$. By Theorem 3.5, the kernel and cokernel of η are p-power torsion, so η is also an equivalence on the $p \neq 0$ locus in the algebraization and hence on U. The complement of U as a substack has everywhere codimension two, as the complement is

$$(\operatorname{Spec}\Gamma(\mathrm{X}_{\mathbf{F}_p}, \mathcal{O})[\![u_0]\!][u]/u_0^n)/\mathbf{G}_m$$

where u has degree -1 and $\deg \mathrm{E}(u_0) = n$.

Passing to the Rees algebra smooth cover $\operatorname{Spec}\mathrm{A}\langle u,t\rangle/(ut-d)$ of the Rees stack, the hypotheses of Stacks Project 0EBJ are satisfied and tell us the pullback to the open $\mathrm{D}(t)\cup\mathrm{D}(p)$ over U on this cover is fully faithful (hence conservative). Since η induces an equivalence on this open, by descent we deduce that η is an equivalence. □

Theorem 3.18. *Let* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *be smooth and quasicompact. The functor* $\mathrm{T}_{\text{ét}}$ *induces an equivalence*

$$\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta).$$

Moreover, if $\mathcal{E} \in \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ *then* $\mathrm{T}_{\text{ét}}(\mathrm{H}^i(\mathcal{E}))[1/p]$ *is crystalline.*

Proof. Apply the main result from [GR24] that $\mathrm{T}^{\Delta}_{\text{ét}}$ induces an equivalence from analytic prismatic F-crystals to $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$. The étale realization for F-gauges factors as

$$\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \xrightarrow{(-)|_{\mathrm{X}^{\Delta}}} \mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$$

and we know now that $(-)|_{\mathrm{X}^{\Delta}}$ has an inverse Π_{X} by the previous theorem.

For the second claim, as $\mathrm{H}^i(\mathcal{E})$ is coherent it suffices to observe that on a smooth affine for a coherent F-gauge $\mathcal{E}$ we have

$$\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p] \simeq \mathrm{T}_{\text{ét}}(\mathcal{E}^{\vee\vee})[1/p]$$

as the étale realization is compatible with duals, and $\mathbf{Q}_p$-local systems are canonically isomorphic to their double dual. As $\mathcal{E}^{\vee\vee}$ is reflexive (the canonical map to the double dual being an isomorphism can be checked on the Noetherian regular cover of $\mathrm{X}^{\mathrm{Syn}}$ given by a covering family of Breuil-Kisin prisms via Proposition 2.9), the claim follows. ☐

We remark that the first equivalence $\mathsf{Refl}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ does not use that X is quasicompact; this is only used for the isogeny part of the statement.

4. PERFECT COMPLEXES UP TO ISOGENY ON $\mathrm{X}^{\mathrm{Syn}}$

4.1. The essential image of $\mathrm{T}_{\text{ét}}$.

We can now apply our results to study the essential image of $\mathrm{T}_{\text{ét}}$ on $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth and quasicompact.

Corollary 4.1. *The t-exact functor*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \xrightarrow{\mathrm{T}_{\text{ét}}[1/p]} \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$$

induces an equivalence $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p] \simeq \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)$ on the heart. The essential image of $\mathrm{T}_{\text{ét}}[1/p]$ contains the essential image of

$$\mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)) \to \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p],$$

and is contained in the full subcategory of $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ where every cohomology sheaf is crystalline.

Proof. The t-exactness of $\mathrm{T}_{\text{ét}}$ follows from Lemma 2.16.

We now show $\mathrm{T}_{\text{ét}}[1/p]$ induces an equivalence on the heart onto the subcategory of crystalline $\mathbf{Q}_p$-local systems, which by Zariski descent of $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$ and $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ can be tested on a smooth affine (here we also use quasicompactness to ensure the covering is finite). The functor $\mathrm{T}_{\text{ét}}$ has the desired essential image on the heart using Theorem 3.18. For full faithfulness, we have a map

$$\mathrm{T}_{\text{ét}} : \mathrm{Hom}_{\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})}(\mathcal{E}, \mathcal{E}')[1/p] \to \mathrm{Hom}(\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p], \mathrm{T}_{\text{ét}}(\mathcal{E}')[1/p]).$$

We use that the map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an equivalence in the isogeny category $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$. Indeed, after étale realization and inverting p it becomes an equivalence, because an isogeny $\mathbf{Z}_p$-local system is isomorphic to its double dual. Thus the kernel and cokernel of the canonical map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ are killed by $\mathrm{T}_{\text{ét}}[1/p]$. Applying Proposition 3.11 and the proof of Theorem 3.5 which upgrades the statement to the F-gauge being p-power torsion, the kernel and cokernel must be 0 in $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$. Thus we can always assume both coherent F-gauges are reflexive. But there we already know $\mathrm{T}_{\text{ét}}$ is an equivalence by Theorem 3.18.

The claim that the essential image is contained in the subcategory of $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ where every cohomology sheaf is crystalline follows immediately from the claim on the heart by t-exactness. For producing objects in the essential image, we use the commutative diagram

$$\begin{tikzcd}
\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}))[1/p] \arrow[r, "\simeq"] \arrow[d] & \mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)) \arrow[d] \\
\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \arrow[r, "\mathrm{T}_{\text{ét}}"] & \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p].
\end{tikzcd}$$

The top arrow is an equivalence coming from the result on the heart and $\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}))[1/p] \simeq \mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p])$, which is true since we work on bounded derived categories. Commutativity of the diagram is easy to verify, as both composite functors to $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ agree on the heart and commute with shifts and fibers (so using [Hau24] Lemma 5.4.3 we can deduce they agree).

Finally, the functor $\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})) \to \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ a priori might not land in $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$ but rather $\mathrm{D}^b_{\mathsf{Coh}}(\mathrm{X}^{\mathrm{Syn}})$, the full subcategory of locally bounded objects in $\mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}})$ where all cohomology sheaves are coherent. However this coincides with $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$, as since X is smooth there is a locally Noetherian regular flat-local surjection ρ such that pullback to the covering is t-exact on $\mathrm{D}_{\mathrm{qc}}(-)$. The t-exactness follows from Remark 2.13. Then given $\mathcal{E} \in \mathrm{D}_{\mathrm{qc}}(\mathrm{X}^{\mathrm{Syn}})$, belonging to $\mathrm{D}^b_{\mathsf{Coh}}(\mathrm{X}^{\mathrm{Syn}})$ means $\mathrm{H}^i(\rho^*\mathcal{E})$ lands in Coh and the complex is locally bounded. But on the cover $\mathrm{D}^b_{\mathsf{Coh}} = \mathsf{Perf}$ and we may test membership in Perf after pullback along a flat-local surjection by descent. Thus the functor is well-defined. We deduce the essential image of $\mathrm{T}_{\text{ét}}$ contains the essential image of $\mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta))$ in $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$. ☐

Remark 4.2. All of the inclusions in Corollary 4.1 are strict in general, although we will later find for a point $\mathrm{X} = \operatorname{Spf} \mathcal{O}_K$ the essential image coincides with the essential image of $\mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K))$ in $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\operatorname{Spa} K, \mathbf{Z}_p)[1/p]$.

In general there are objects in $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ which have crystalline cohomologies but are not in the essential image of $\mathrm{T}_{\text{ét}}$. For example, on a point $\mathrm{X}_\eta = \operatorname{Spa} K$, on the target $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\operatorname{Spa} K, \mathbf{Z}_p)[1/p]$ there exists a nonzero map $\mathbf{Q}_p \to \mathbf{Q}_p(1)[2]$ using that the second rationalized étale cohomology of $\mathbf{Z}_p(1)$ is $\mathbf{Q}_p$. Taking the fiber of this map, we obtain a complex with crystalline cohomology sheaves which cannot be obtained from $\mathrm{T}_{\text{ét}}$ (one may see this by computing $\mathrm{H}^2_{\mathrm{Syn}}(\mathcal{O}_K, \mathcal{O}\{1\})[1/p] = 0$ so there is no corresponding map of F-gauges and using full faithfulness on the heart after inverting p). However, as we will see in Proposition 4.14 in the case of a point the essential image is exactly the essential image of $\mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K))$ in $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\operatorname{Spa} K, \mathbf{Z}_p)[1/p]$.

There are also objects in the essential image of $\mathrm{T}_{\text{ét}}$, but not contained in the essential image of $\mathrm{D}^b(\mathsf{Loc}^{\text{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)) \to \mathrm{D}^{(b)}_{\text{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ when $\mathrm{X} = \mathbf{P}^1_{\mathcal{O}_K}$. Using Example 4.7 we can produce a map $f : \mathcal{O} \to \mathcal{O}\{1\}[2]$ in $\mathsf{Perf}(\mathrm{X}^{\text{Syn}})$ so that $\mathrm{T}_{\text{ét}}(\text{fib}(f))$ fails to come from $\mathrm{D}^b(\mathsf{Loc}^{\text{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta))$ as Ext^2 for objects in the heart vanishes for this category when $\mathrm{X} = \mathbf{P}^1_{\mathcal{O}_K}$.

We may also use this to identify Ext^1 groups in crystalline $\mathbf{Q}_p$-local systems with rationalized syntomic cohomology.

Proposition 4.3. *Assume* X *is smooth and quasicompact over* $\mathrm{Spf}\ \mathcal{O}_K$ *and let* $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\text{Syn}})$. *Then the étale realization induces an isomorphism*

$$\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E})[1/p] \simeq \mathrm{Ext}^1_{\mathsf{Loc}^{\text{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)}(\mathbf{Q}_p, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]).$$

In particular, if $\mathrm{X} = \mathrm{Spf}\ \mathcal{O}_K$ *then* $\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E})[1/p] = \mathrm{H}^1_f(\mathrm{G}_K, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p])$ *where* H^1_f *denotes the Bloch-Kato Selmer group.*

Proof. Due to $\mathrm{T}_{\text{ét}}$ inducing an equivalence $\mathsf{Coh}(\mathrm{X}^{\text{Syn}})[1/p] \simeq \mathsf{Loc}^{\text{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)$ as shown in Corollary 4.1, we immediately have

$$\mathrm{Ext}^i_{\mathsf{Coh}(\mathrm{X}^{\text{Syn}})[1/p]}(\mathcal{E}[1/p], \mathcal{E}'[1/p]) = \mathrm{Ext}^i_{\mathsf{Loc}^{\text{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta)}(\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p], \mathrm{T}_{\text{ét}}(\mathcal{E}')[1/p])$$

for coherent F-gauges $\mathcal{E}$ and $\mathcal{E}'$ as an equivalence of abelian categories gives an equivalence of their Yoneda Ext groups.

It then suffices to prove the stronger integral claim that

$$\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E}) \simeq \mathrm{Ext}^1_{\mathsf{Coh}(\mathrm{X}^{\text{Syn}})}(\mathcal{O}, \mathcal{E}).$$

We will first argue

$$\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E}) \simeq \mathrm{Ext}^1_{\mathsf{QCoh}(\mathrm{X}^{\text{Syn}})}(\mathcal{O}, \mathcal{E})$$

where the Ext^1 group here is a Yoneda Ext group (so it makes sense in any abelian category). This is easy to show in general, since we may identify $\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathrm{D}_{\text{qc}}(\mathrm{X}^{\text{Syn}})}(\mathcal{O}, \mathcal{E}[1])$ and then the latter classifies extensions: we map an extension to the induced connecting morphism, and a map $\mathcal{O} \to \mathcal{E}[1]$ to its fiber.

Given an extension in $\mathsf{QCoh}(\mathrm{X}^{\text{Syn}})$ of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{F}$ and $\mathcal{G}$ are coherent we know that $\mathcal{E}$ is coherent. We can test this after pullback to a regular Noetherian flat cover, at which point it follows from the 2 out of 3 property for exact sequences of coherent sheaves. This then means $\mathrm{H}^1_{\text{Syn}}(\mathrm{X}, \mathcal{E}) \simeq \mathrm{Ext}^1_{\mathsf{Coh}(\mathrm{X}^{\text{Syn}})}(\mathcal{O}, \mathcal{E})$. ☐

Next, we will study what happens to both sides of Corollary 4.1 under smooth proper pushforwards. We will need the following lemma to deal with pullbacks.

Lemma 4.4. *The pullbacks j^*_{Nyg} and $j^*_{\mathbb{\Delta}}$ on $\mathrm{D}^b_{\mathrm{qc}}((-)^{\mathrm{Syn}})$ commute with pushforwards of F-gauges along maps of quasisyntomic p-adic formal schemes.*

Proof. First consider the commutative diagram

$$\begin{tikzcd}
\mathrm{X}^{\mathbb{\Delta}} \arrow[r, "j_{\mathbb{\Delta}}"] \arrow[d, "f"] & \mathrm{X}^{\mathrm{Syn}} \arrow[d, "f"] \\
\mathrm{Y}^{\mathbb{\Delta}} \arrow[r, "j_{\mathbb{\Delta}}"] & \mathrm{Y}^{\mathrm{Syn}}
\end{tikzcd}$$

where $j_{\mathbb{\Delta}}$ is an open immersion. For the second claim, we want to know the canonical natural transformation $j^*_{\mathbb{\Delta}}\mathrm{R}f_* \to \mathrm{R}f_*j^*_{\mathbb{\Delta}}$ is an equivalence when we restrict to coefficients in the bounded derived category of quasicoherent sheaves (see Definition A.20 in [AKN23]).

The claim that $j^*_{\mathbb{\Delta}}\mathrm{R}f_* \simeq \mathrm{R}f_*j^*_{\mathbb{\Delta}}$ follows purely formally, by applying Lemma A.28 in [AKN23] after using quasisyntomic descent on the relevant stacks to write them as colimits. This reduces us to showing base change for the diagram

$$\begin{tikzcd}
\mathrm{Spf}\,\mathbb{\Delta}_{\mathrm{R}} \arrow[r, "j_{\mathbb{\Delta}}"] \arrow[d, "f"] & \mathrm{R}^{\mathrm{Syn}} \arrow[d, "f"] \\
\mathrm{Spf}\,\mathbb{\Delta}_{\mathrm{R}'} \arrow[r, "j_{\mathbb{\Delta}}"] & (\mathrm{R}')^{\mathrm{Syn}}
\end{tikzcd}$$

for a map of quasiregular semiperfectoid rings $\mathrm{Spf}\,\mathrm{R} \to \mathrm{Spf}\,\mathrm{R}'$ (the edge base change squares required in this case are trivial to verify). The analogous square with the Nygaard stack replacing $\mathrm{R}^{\mathrm{Syn}}$ and $(\mathrm{R}')^{\mathrm{Syn}}$ has base change for both of the open immersions j_{HT} and j_{dR}, so again applying Lemma A.28 in [AKN23] for the diagram $\begin{tikzcd} \ast \arrow[d] & \ast \arrow[r] & \ast \\ \ast & & \end{tikzcd}$ used to produce $\mathrm{X}^{\mathrm{Syn}}$ as a pushout of formal stacks shows commutativity of the left square.

An identical argument shows the claim for j^*_{Nyg}. We reduce to base change for the diagram

$$\begin{tikzcd}
\mathrm{R}^{\mathrm{Nyg}} \arrow[r, "j_{\mathrm{Nyg}}"] \arrow[d, "f"] & \mathrm{R}^{\mathrm{Syn}} \arrow[d, "f"] \\
(\mathrm{R}')^{\mathrm{Nyg}} \arrow[r, "j_{\mathrm{Nyg}}"] & (\mathrm{R}')^{\mathrm{Syn}}
\end{tikzcd}$$

which we may check in the same way. ☐

We then have the following (already known) pushforward stability result. This appears in [MM25] Proposition 8.2.6; below we explain some of the argument.

Proposition 4.5 ([MM25] Proposition 8.2.6)**.** *Let $f : \mathrm{X} \to \mathrm{Y}$ be a smooth proper morphism of smooth p-adic formal schemes over* $\operatorname{Spf} \mathcal{O}_K$*. Then if* $\mathcal{E} \in \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$*, we have* $\mathrm{R}f_*\mathcal{E} \in \mathsf{Perf}(\mathrm{Y}^{\mathrm{Syn}})$.

Proof. It suffices to check this for $(-)^{\mathrm{Nyg}}$ using flat descent of Perf, as $\mathrm{Y}^{\mathrm{Nyg}}$ is an étale cover of $\mathrm{Y}^{\mathrm{Syn}}$ and similarly for $\mathrm{X}^{\mathrm{Nyg}}$ and $\mathrm{X}^{\mathrm{Syn}}$ (see the discussion in [Bha22] Definition 6.1.1). Then we may use that j^*_{Nyg} commutes with pushforwards as verified in the previous lemma to reduce to checking $\mathrm{R}f_*$ sends perfect complexes on $\mathrm{X}^{\mathrm{Nyg}}$ to perfect complexes on $\mathrm{Y}^{\mathrm{Nyg}}$ for smooth proper morphisms f. There is a stratification of $\mathrm{X}^{\mathrm{Nyg}}$ by $\mathrm{X}^{\mathrm{Nyg}}_{t=0}$ and $\mathrm{X}^{\mathrm{Nyg}}_{t\neq 0}$, as well as $\mathrm{Y}^{\mathrm{Nyg}}$.

Now we observe that it suffices to check perfectness on the $t = 0$ component of $\mathrm{Y}^{\mathrm{Nyg}}$ and derived t-completeness. One may also use formal gluing for this stratification to make the reduction by checking perfectness is also preserved for the pushforward $\mathrm{X}^{\mathrm{Nyg}}_{t\neq 0} \simeq \mathrm{X}^{\mathbb{\Delta}} \to \mathrm{Y}^{\mathrm{Nyg}}_{t\neq 0} \simeq \mathrm{Y}^{\mathbb{\Delta}}$, which is well-known (see e.g. Corollary 5.16 in [GR24]). The pushforward $\mathrm{R}f_*\mathcal{E}$ will still be complete for the Nygaard filtration as one can reduce to the same assertion for Hodge-filtered de Rham cohomology in the smooth proper case by the same method as in [BL22a] Proposition 5.8.2. The claim for the $t = 0$ component is shown in [MM25], via reduction to the case where Y is semiperfectoid and then reducing the claim to smooth proper pushforwards of schemes preserving perfect complexes. □

This then formally implies the following result.

Corollary 4.6. *Suppose* X *and* Y *are smooth over* $\operatorname{Spf} \mathcal{O}_K$ *and let* $\mathbb{L} \in \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)[1/p]$ *be in the essential image of* $\mathrm{T}_{\text{ét}}$ *(as partially characterized in Corollary 4.1). Then if* $f : \mathrm{X} \to \mathrm{Y}$ *is smooth proper,* $\mathrm{R}^i f_*\mathbb{L}$ *is crystalline for all* i.

Proof. It suffices to show that $\mathrm{T}_{\text{ét}}$ commutes with smooth proper pushforwards in light of Corollary 4.1. In the diagram

$$\begin{tikzcd}
\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \arrow[r, "(-)|_{\mathrm{X}^{\mathbb{\Delta}}}"] \arrow[d, "\mathrm{R}f_*"] & \mathsf{Perf}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \arrow[r] \arrow[d, "\mathrm{R}f_*"] & \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p) \arrow[d, "\mathrm{R}f_*"] \\
\mathsf{Perf}(\mathrm{Y}^{\mathrm{Syn}}) \arrow[r, "(-)|_{\mathrm{Y}^{\mathbb{\Delta}}}"] & \mathsf{Perf}^{\varphi}(\mathrm{Y}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \arrow[r] & \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Y}_\eta, \mathbf{Z}_p)
\end{tikzcd}$$

we know right hand square commutes by Theorem 5.8 in [GL23], so it suffices to verify the left square commutes and $\mathrm{R}f_*$ preserves perfect F-gauges. The latter holds by Proposition 4.5.

Now we check commutativity of the left square. For prismatic F-crystals, by an inspection of the canonical prismatic F-crystal structure given to $j^*_{\mathbb{\Delta}}(\mathcal{E})$ in Lemma 2.3 it suffices to know j^*_{Nyg} and $j^*_{\mathbb{\Delta}}$ commute with pushforwards, which we have already shown in Lemma 4.4. □

The result that $\mathrm{R}^i f_* \mathbb{L}[1/p]$ is crystalline when $\mathbb{L} \in \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{X}_\eta)$ and f is smooth proper is given by Theorem B [GR24] (the isocrystal association portion of the statement is verified later in Proposition 4.22). We can regard this as strengthening the result to allow more general $\mathbb{L}$ in the derived category. When $\mathrm{Y} = \mathrm{Spf}\, \mathcal{O}_K$, by Proposition 4.14 we actually learn about the complex beyond its cohomology, namely that it lies in the essential image of $\mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)) \to \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spa}\, K, \mathbf{Z}_p)[1/p]$.

4.2. Admissible filtered F-isocrystals in the proper case.

In Proposition 5.6.2 of [Hau24], Corollary 4.1 is refined to an equivalence in the case of $\mathrm{X} = \mathrm{Spf}\, \mathbf{Z}_p$. In particular, it is shown that $\mathrm{T}_{\text{ét}}$ induces an equivalence

$$\mathsf{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_{\mathbf{Q}_p})).$$

Such an equivalence cannot hold in general, as the following example shows.

Example 4.7. *Let* X *be the* p*-adic formal scheme* $\mathbf{P}^1_{\mathbf{Z}_p}$*. If we had an equivalence*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}((\mathbf{P}^1_{\mathbf{Q}_p})^{\mathrm{ad}}))$$

then considering $\mathrm{RHom}_{(\mathbf{P}^1_{\mathbf{Z}_p})^{\mathrm{Syn}}}(\mathcal{O}, \mathcal{O}\{1\})[1/p] \simeq \mathrm{R}\Gamma_{\mathrm{Syn}}(\mathbf{P}^1_{\mathbf{Z}_p}, \mathcal{O}\{1\})[1/p]$ *we would need to show that*

$$\mathrm{H}^i_{\mathrm{Syn}}(\mathbf{P}^1_{\mathbf{Z}_p}, \mathcal{O}\{1\})[1/p] \simeq \mathrm{Ext}^i_{\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}((\mathbf{P}^1_{\mathbf{Q}_p})^{\mathrm{ad}})}(\mathbf{Q}_p, \mathbf{Q}_p(1)).$$

When $i = 2$*, the right hand side must vanish, since* Ext^2 *vanishes in crystalline* $\mathbf{Q}_p$*-local systems. Indeed, since* $(\mathbf{P}^1_{\mathbf{Q}_p})^{\mathrm{ad}}$ *has an étale fundamental group isomorphic to* $\mathrm{G}_{\mathbf{Q}_p}$*, the subcategory of crystalline local systems is equivalent to the category of crystalline Galois representations of* $\mathrm{G}_{\mathbf{Q}_p}$*. In this abelian category,* Ext^2 *will vanish (by* [EK99]*; see also* [Hau24] *Remark 5.2.5).*

However the left hand side (syntomic cohomology) is nonzero when $i = 2$*, as we get a nonzero class coming from the syntomic Chern class (see Variant 8.4.5 in* [BL22a]*).*

In this subsection we will first show that despite this failure in the higher dimensional case, Hauck's result still extends to the case where $K/\mathbf{Q}_p$ is an arbitrary finite extension:

$$\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)).$$

We will then define a category $\mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^\varphi}(\mathrm{X})$ of admissible filtered F-isocrystals in perfect complexes and use the case of $\mathcal{O}_K$ to show that there is a fully faithful functor

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \to \mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^\varphi}(\mathrm{X})$$

when $\mathrm{X}/\mathrm{Spf}\, \mathcal{O}_K$ is smooth proper, and then use §4.1 to see the functor is essentially surjective.

The key input in the case of a point is an analogue of the cohomology computation done in [Bha22] Proposition 6.7.3 for arbitrary finite extensions $K/\mathbf{Q}_p$.

Proposition 4.8. *Let $K/\mathbf{Q}_p$ be a finite extension of $\mathbf{Q}_p$. Then for $\mathcal{E} \in \mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ we have*

$$\mathrm{H}^i(\mathcal{O}_K^{\mathrm{Syn}}, \mathcal{E})[1/p] \simeq \mathrm{Ext}^i_{\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)}(\mathbf{Q}_p, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p])$$

where the 0th extension group on the right is $\mathrm{H}^0(\mathrm{G}_K, (-)[1/p])$, the first is given by

$$\mathrm{H}^1_f(\mathrm{G}_K, -) = \mathrm{Ext}^1_{\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)}(\mathbf{Q}_p, (-)[1/p]),$$

and both sides vanish for $i \geq 2$.

Assuming the conjectured Lagrangian refinement of Tate duality for regular formal schemes, see Conjecture 6.5.25 in [Bha22], one may deduce this using the same strategy as [Bha22] Proposition 6.7.3. If K is unramified, since $\mathrm{Spf}\,\mathrm{W}(k)/\,\mathrm{Spf}\,\mathbf{Z}_p$ is finite étale and in particular smooth proper we may use geometric Poincaré duality to deduce the claim (see the remark following Conjecture 6.5.25 in [Bha22]).

We will use a different approach to prove the claim by adapting the tools of [AKN24] to work with coefficients. As a byproduct, we will produce a complex computing syntomic cohomology using Breuil-Kisin modules.

Notation. In what follows we will abbreviate absolute prismatic cohomology $\mathrm{R\Gamma}(\mathcal{O}_K^{\mathbb{\Delta}}, \mathcal{E})$ as $\mathbb{\Delta}_{\mathcal{O}_K}(\mathcal{E})$ and the relative prismatic cohomology $\mathrm{R\Gamma}((\mathcal{O}_K/\mathrm{A})^{\mathbb{\Delta}}, \mathcal{E})$ as $\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{A}}(\mathcal{E})$ for a prismatic crystal $\mathcal{E}$.

Using [AKN23] Theorem 1.2(6) with coefficients (which follows from the identification of stacks in Corollary 2.11 after restricting to the $t \neq 0$ locus), we may compute

$$\mathbb{\Delta}_{\mathcal{O}_K}(\mathcal{E}) \simeq \mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)}(\mathcal{E})$$

as

$$\mathrm{Tot}\left(\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \rightrightarrows \mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}(\mathcal{E}) \Rrightarrow \cdots \right)$$

giving a descent complex

$$\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \xrightarrow{\mathrm{d}^1} \mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}(\mathcal{E}) \xrightarrow{\mathrm{d}^2} \dots$$

The analogous descent complex also computes Nygaard filtered prismatic cohomology after endowing each $\mathbb{\Delta}^{(1)}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,\dots,u_i]\!]}(\mathcal{E})$ with the naïve Nygaard filtration for $\mathrm{I} = \mathrm{E}(u_0)$ (as $\mathcal{O}_K$ is relatively quasisyntomic in the sense of [AKN23]). We can use Corollary 2.11 to formally justify this construction with coefficients. We can use this along with a result of Gao-Min-Wang to obtain a shorter description of prismatic cohomology.

Lemma 4.9. *Let $\mathcal{E} \in \mathsf{Vect}(\mathcal{O}_K^{\mathbb{\Delta}})$. Then*

$$\mathrm{R\Gamma}(\mathcal{O}_K^{\mathbb{\Delta}}, \mathcal{E}) \simeq \left[\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \to K\right].$$

where $K = \ker(\mathrm{d}^2 : \mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}(\mathcal{E}) \to \mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1,u_2]\!]}(\mathcal{E}))$ is the kernel of the second differential in the descent complex.

Proof. This follows from the result of Gao-Min-Wang in [GMW23] that $\mathrm{R\Gamma}(\mathcal{O}_K^{\mathbb{\Delta}}, \mathcal{E})$ lies in cohomological degrees 0 and 1. ☐

In what follows, all filtrations F^* on rings have the property that $\mathrm{F}^0\mathrm{A} \simeq \mathrm{F}^{-n}\mathrm{A}$ for $n \geq 0$ and $\mathrm{F}^0\mathrm{A} = \mathrm{A}$. A filtered δ-pair consists of a filtered δ-ring $\mathrm{F}^*\mathrm{A}$ (one requires $\delta(\mathrm{F}^n\mathrm{A}) \subseteq \mathrm{F}^{pn}\mathrm{A}$) and a map of filtered rings $\mathrm{F}^*\mathrm{A} \to \mathrm{F}^*\mathrm{R}$. In [AKN23] §10 a filtered variant of prismatic cohomology $\mathrm{F}^*{\mathbbl{\Delta}}_{\mathrm{R}/\mathrm{A}}$ is given relative to filtered δ-pairs $\mathrm{F}^*\mathrm{A} \to \mathrm{F}^*\mathrm{R}$. From now on we give the Breuil-Kisin prism $\mathrm{W}(k)[\![u_0]\!]$ the u_0-adic filtration, and $\mathcal{O}_K$ the π-adic filtration for a uniformizer π. For a filtered δ-pair $\mathrm{F}^*\mathrm{A} \to \mathrm{F}^*\mathrm{R}$, the relative prismatic cohomology inherits a filtration $\mathrm{F}^*{\mathbbl{\Delta}}_{\mathrm{R}/\mathrm{A}}$. With coefficients in a prismatic crystal $\mathcal{E}$, we let $\mathrm{F}^*{\mathbbl{\Delta}}_{\mathrm{R}/\mathrm{A}}(\mathcal{E})$ be induced from the filtration $\mathrm{F}^*{\mathbbl{\Delta}}_{\mathrm{R}/\mathrm{A}}$ (the flat filtration in the terminology of [AKN23]). Write $\mathrm{F}^*\widehat{{\mathbbl{\Delta}}}_{\mathrm{R}/\mathrm{A}}$ for the completion with respect to this filtration.

We will need a $\mathrm{W}(k)[\![u_0]\!]$-linear map

$$\Theta : \mathrm{F}^*\varphi^*\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}(\mathcal{E}) \to \varphi^*\mathrm{F}^{*-1}\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}\{-1\})$$

as well as a Nygaard filtered version, which shifts the Nygaard filtration down by one. In [AKN24] this is constructed for $\mathcal{E} = \mathcal{O}_{\mathbbl{\Delta}}$, which induces the general map. It is helpful to understand this construction via the perspective of comodules over a Hopf algebroid, which we now explain.

In §4.6 of [AKN24] they view prismatic crystals as right comodules over a Hopf algebroid $(\Gamma_0, \Gamma_1) = (\mathrm{F}^*\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}, \mathrm{F}^*\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]})$ corresponding to the truncated F-completed descent complex. This is only used for Breuil-Kisin twists, but the same general construction applies verbatim in our setting for prismatic crystals in vector bundles. There is of course also a Frobenius twisted variant deduced from the untwisted one, instead using the Hopf algebroid $(\mathrm{F}^*\varphi^*\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}, \mathrm{F}^*\varphi^*\widehat{{\mathbbl{\Delta}}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]})$. By using the g_u basis of Proposition 3.34 for Γ_1 constructed in [AKN24], they give a map $\Theta : \Gamma_1 \to \Gamma_0$ in Definition 4.41 in both the Frobenius twisted and untwisted cases. This induces a connection ∇_Θ on an arbitrary right comodule over this Hopf algebroid via the setup of §4.6 as the composite $\mathrm{M} \xrightarrow{\rho} \mathrm{M} \otimes_{\Gamma_0} \Gamma_1 \xrightarrow{1\otimes\Theta} \mathrm{M}$ where ρ is the coaction and the second map identifies with the Θ we needed previously in the Frobenius twisted case (we trivialize the Breuil-Kisin twist for convenience by picking a generator $\mathrm{E}(u_0)$ for the prismatic ideal). We may also use the same argument as in Lemma 4.36 in [AKN24] to identify the descent complex of a prismatic crystal in vector bundles with the cobar complex of the associated comodule.

There is also a Nygaard filtered version, as by virtue of Corollary 2.11 we may view sheaves on the stack $\mathcal{O}_K^{\mathrm{Nyg}}$ as Nygaard filtered right comodules for the Frobenius twisted variant $(\Gamma_0^{(1)}, \Gamma_1^{(1)})$; the map Θ is Nygaard filtered, so it upgrades to a map of Rees stacks once we apply F-completion to each Rees stack. We will see later F-completion is automatic for Γ_0 which is useful (in particular allowing ∇_M to exist before completion), but we must still involve it in the construction as it is necessary for Θ to be defined. Thus to describe coefficients for Nygaard filtered F-completed prismatic cohomology as right comodules we apply F-completion to each Rees stack used in the descent complex to obtain an F-completed Nygaard filtered variant. The same remark about the cobar complex and descent complex matching again holds, now with Nygaard filtrations.

By the previous discussion we already have a connection $\nabla_{\mathcal{E}}$ on $\varphi^*\mathrm{F}^*\widehat{\mathbb{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E})$. There is also an equivalent formulation using the descent complex, which will be convenient to have for proofs.

Definition 4.10 ([AKN24]). *Define*

$$\nabla_{\mathcal{E}} : \mathrm{F}^*\widehat{\mathbb{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \to \mathrm{F}^{*-1}\widehat{\mathbb{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}\{-1\})$$

to be the composition of Θ *and the first differential of the descent complex, and similarly we may define Frobenius twisted and Nygaard filtered variants (recall we use Corollary 2.11 to make sense of coefficients). We will trivialize this Breuil-Kisin twist by picking a generator* $\mathrm{E}(u_0)$ *for the prismatic ideal in* $\mathrm{W}(k)[\![u_0]\!]$.

Using Proposition 10.41(c) in [AKN23] we deduce the source and target of $\nabla_{\mathcal{E}}$ are already F-complete, allowing us to drop the completion. We can now check that a variant of Corollary 4.44 in [AKN24] with coefficients holds. The map φ^{∇} appearing in the statement will be clarified in the proof.

Lemma 4.11. *Let* $\mathcal{E}$ *be a reflexive* F*-gauge on* $\mathcal{O}_K^{\mathrm{Syn}}$*, and pick a Breuil-Kisin prism* $(\mathrm{W}(k)[\![u_0]\!], \mathrm{E}(u_0))$ *for* $\mathcal{O}_K$*. Let* M *be the associated Breuil-Kisin module to* $\mathcal{E}$ *with the naïve Nygaard filtration* $\mathrm{Fil}^i_{\mathrm{Nyg}}\varphi^*\mathrm{M} = \{x \in \varphi^*\mathrm{M} : \varphi_{\mathrm{M}}(x) \in \mathrm{E}(u_0)^i\mathrm{M}\}$*, and let* $\mathrm{F}^*\mathrm{M}$ *be the filtration induced by regarding* $\mathrm{W}(k)[\![u_0]\!]$ *as a filtered* δ*-ring where* $\deg(u_0) = 1$.

Then $\mathrm{R}\Gamma_{\mathrm{Syn}}(\mathcal{O}_K, \mathcal{E})$ *can be computed as the total fiber of the square*

$$\begin{tikzcd}
\mathrm{F}^*\mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*\mathrm{M} \arrow[r, "\mathrm{Fil}^0_{\mathrm{Nyg}}\nabla_{\mathrm{M}}"] \arrow[d, "\mathrm{can}-\varphi_{\mathrm{M}}"] & \mathrm{F}^{*-1}\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\mathrm{M} \arrow[d, "\mathrm{can}-\varphi^{\nabla}_{\mathrm{M}}"] \\
\mathrm{F}^*\varphi^*\mathrm{M} \arrow[r, "\nabla_{\mathrm{M}}"] & \mathrm{F}^{*-1}\varphi^*\mathrm{M}
\end{tikzcd}$$

for a map φ^{∇}.

Proof. This follows from the same method as Corollary 4.44 in [AKN24] but with minor modifications for coefficients. Let $\mathrm{M} = \mathrm{R}\Gamma((\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!])^{\mathbb{\Delta}}, \mathcal{E})$ be the Breuil-Kisin module attached to $\mathcal{E}$. Using Lemma 4.9 we see

$$\mathrm{R}\Gamma(\mathcal{O}_K^{\mathbb{\Delta}}, \mathcal{E}) \simeq \left[\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \to K_2\right]$$

where K_2 is the kernel of the second differential in the descent complex. Similarly, we have

$$\mathrm{R}\Gamma(\mathcal{O}_K^{\mathrm{Nyg}}, \mathcal{E}) \simeq \left[\mathrm{Fil}^0_{\mathrm{Nyg}}\,\varphi^*\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \to \mathrm{Fil}^0_{\mathrm{Nyg}}\,\varphi^*K_2\right].$$

Thus, the syntomic cohomology of the reflexive F-gauge $\mathcal{E}$ is given by the total fiber of

$$\begin{tikzcd}
\mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*\mathrm{M} \arrow[r] \arrow[d, "\mathrm{can}-\varphi_{\mathrm{M}}"] & \mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*K_2 \arrow[d, "\mathrm{can}-\varphi_{K_2}"] \\
\varphi^*\mathrm{M} \arrow[r] & \varphi^*K_2
\end{tikzcd}$$

and the horizontal maps come from the first differential. We can take F-completions on every term, since syntomic cohomology with coefficients in a reflexive F-gauge is still F-complete. Indeed, the proof of Proposition 2.11 in [AKN24] readily generalizes as Proposition 8.6 in [AKN23] works with coefficients (syntomic cohomology with coefficients lands in their category of quadruples (H, N, c, φ)).

Thus, we see syntomic cohomology is computed as the total fiber of the F-completed square

$$\begin{tikzcd}
\mathrm{F}^*\mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*\widehat{\mathrm{M}} \arrow[r] \arrow[d, "\mathrm{can}-\varphi_{\mathrm{M}}"] & \mathrm{F}^*\mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*\widehat{K_2} \arrow[d, "\mathrm{can}-\varphi_{K_2}"] \\
\mathrm{F}^*\varphi^*\widehat{\mathrm{M}} \arrow[r] & \mathrm{F}^*\varphi^*\widehat{K_2}
\end{tikzcd}$$

As before, the symbol $\widehat{\cdot}$ denotes F-completion.

The map Θ is an isomorphism when restricted to K_2, which can be shown by using the same proof as Lemma 4.43 in [AKN24]. For completeness, we give a brief overview of the argument. We see $\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E})$ is a comodule over the Hopf algebroid

$$\Gamma = (\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}, \widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]})$$

and the cobar complex of $\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E})$ identifies with its descent complex. The claim we want to show is that we get a quasi-isomorphism of complexes

$$\begin{tikzcd}
\mathrm{F}^*\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \arrow[r] \arrow[d, equal] & \mathrm{F}^*\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}(\mathcal{E}) \arrow[r] \arrow[d, "\Theta"] & \dots \\
\mathrm{F}^*\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \arrow[r, "\nabla"] & \mathrm{F}^*\widehat{\mathbbl{\Delta}}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E}) \arrow[r] & 0
\end{tikzcd}$$

where the top complex is the F-completed descent complex. The argument of Lemma 4.43 now follows identically by using the cobar resolution of $\mathbbl{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}(\mathcal{E})$ and that $\mathcal{E}$ is a prismatic crystal in vector bundles. The Frobenius twisted and Nygaard filtered versions follow similarly.

Translating the previous square along Θ, one then obtains a square

$$\begin{tikzcd}
\mathrm{F}^*\mathrm{Fil}^0_{\mathrm{Nyg}}\varphi^*\widehat{\mathrm{M}} \arrow[r, "\nabla_{\mathrm{M}}"] \arrow[d, "\mathrm{can}-\varphi_{\mathrm{M}}"] & \mathrm{F}^{*-1}\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\widehat{\mathrm{M}} \arrow[d, "\mathrm{can}-\varphi^{\nabla}_{\mathrm{M}}"] \\
\mathrm{F}^*\varphi^*\widehat{\mathrm{M}} \arrow[r, "\nabla_{\mathrm{M}}"] & \mathrm{F}^{*-1}\varphi^*\widehat{\mathrm{M}}
\end{tikzcd}$$

whose total fiber computes syntomic cohomology as desired. As a final step, we observe that actually $\widehat{\mathrm{M}} = \mathrm{M}$. The F-filtration on $\mathbb{\Delta}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0]\!]}$ is the u_0-adic filtration by Proposition 10.41(c) in [AKN23], so this is already complete; since we work with prismatic crystals in vector bundles and use the induced flat filtration, the desired completeness follows. $\square$

As seen in the proof, the map φ^∇ is induced by the map φ_{M} on the kernel K_2 of d^2 in the descent complex for the prismatic F-crystal, which is identified with the desired map by applying Θ and trivializing the twist. By construction, it satisfies the relation $\nabla \circ \varphi = \varphi^\nabla \circ \nabla$.

We may now use the ramified variant of this result in Lemma 4.11 to deduce the desired vanishing statement. Before proceeding, we note that the connection ∇_{M} does not actually have a Leibniz rule, but it does have one on the associated F-graded with correction terms. In particular, all we will use is that

$$\nabla_{\mathrm{M}}(u_0 m) = \mathrm{C}_i m + n$$

where if $m \in \mathrm{F}^{i-1}\varphi^*\mathrm{M} \setminus \mathrm{F}^i\varphi^*\mathrm{M}$ we may pick $n \in \mathrm{F}^i\varphi^*\mathrm{M}$ and C_i is multiplication by i up to units. In general by writing $u_0 m = u_0^i m_0$ where m_0 is in $\mathrm{F}^0\varphi^*\mathrm{M}$ by definition

$$\nabla_{\mathrm{M}}(u_0^i m_0) = (\mathrm{id} \otimes \Theta)(\rho(u_0^i m_0))$$

where $\rho : \varphi^*\mathrm{M} \to \varphi^*\mathrm{M} \otimes_{\mathrm{W}(k)[\![u_0]\!]} \widehat{\mathbb{\Delta}}^{(1)}_{\mathcal{O}_K/\mathrm{W}(k)[\![u_0,u_1]\!]}$ is the coaction of the comodule $\varphi^*\mathrm{M}$. Here, since we work with the Frobenius twist we clarify that u_0 is $1 \otimes u_0$. We can write the coaction as $\rho(m_0) = m_0 \otimes 1 + \rho_{\geq 1}$, where $\rho_{\geq 1}$ is the contribution in higher F-degrees; the first component is forced by the relation $(\mathrm{id} \otimes \varepsilon)\rho(m_0) = m_0$. That is, $\rho(m_0) - m_0 \otimes 1 \in \ker(\mathrm{id} \otimes \varepsilon)$ and this kernel will lie in $\mathrm{F}^{\geq 1}$ (as $\varepsilon : \Gamma_1 \to \Gamma_0$ sends u_1, u_0 to u_0 and the F-filtration on $\mathrm{W}(k)[\![u_0, u_1]\!] \subset \Gamma_1$ is determined by Proposition 10.41(c) in [AKN23]). Using this, up to higher terms in the F-filtration (i.e. modulo $\mathrm{F}^i\varphi^*\mathrm{M}$) we can compute $(\mathrm{id} \otimes \Theta)(\rho(u_0^i m_0)) = (\mathrm{id} \otimes \Theta)(\eta_R(u_0^i)(m_0 \otimes 1))$. Using the same argument of Lemma 4.61(1) in [AKN24], there is a total contribution of $\mathrm{C}_i m$ where C_i is i up to units to the $\mathrm{gr}^{i-1}_{\mathrm{F}}$ component. That is, working in the twist $\varphi^*\mathrm{M}$ we may expand $(\mathrm{id} \otimes \Theta)(\eta_R(u_0^i)(m_0 \otimes 1))$ as $m_0 \otimes i u_0^{i-1} + m_0 \otimes \sum_{j=2}^{i} \binom{i}{j} u_0^{i-j} \Theta(g_0^j)$ where $g_0 := u_1 - u_0$ (in [AKN24] the map Θ is $\mathrm{W}(k)[\![u_0]\!]^{(1)}$-linear and is defined by what it does on monomials in the g_u basis, starting with g_0 as the first nontrivial one). The same argument allows us to deduce the surviving $j \geq 2$ terms are multiples of $m_0(p \cdot i u_0^{i-1})$, from which the claim follows as they can only change the scalar C_i by a unit. Note that we have already trivialized the Breuil-Kisin twist in this formulation, as we regard ∇_{M} as a map $\varphi^*\mathrm{M} \to \varphi^*\mathrm{M}$.

Proposition 4.12. *If $\mathcal{E}$ is in* $\mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$, *then* $\tau^{\geq 2}\mathrm{R}\Gamma(\mathcal{O}_K^{\mathrm{Syn}}, \mathcal{E})[1/p] = 0$.

Proof. Without loss of generality, we may assume that $\mathcal{E}$ is reflexive: up to p-isogeny this is always possible by the proof of Corollary 4.1. Indeed, replacing $\mathcal{E}$ by its reflexive hull $\mathcal{E}^{\vee\vee}$ does not change the claim, since the kernel and cokernel of $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ are p-power torsion, and hence their syntomic cohomology is p-power torsion as well.

By Lemma 4.11 it suffices to check that the map

$$\mathrm{F}^{*-1}\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\mathrm{M} \oplus \mathrm{F}^*\varphi^*\mathrm{M} \xrightarrow{\begin{pmatrix} \mathrm{can}-\varphi^{\nabla}_{\mathrm{M}} \\ \nabla_{\mathrm{M}} \end{pmatrix}^{\mathrm{T}}} \mathrm{F}^{*-1}\varphi^*\mathrm{M}$$

is surjective after inverting p. Call this map α.

Assume first that $\mathcal{E}$ has all Hodge–Tate weights ≥ 0. Then $\varphi^{\nabla}_{\mathrm{M}}$ is defined integrally on $\varphi^*\mathrm{M}$, and M is u_0-adically complete. In the effective case, $\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\mathrm{M}[1/p] \subseteq \varphi^*\mathrm{M}[1/p]$ is an equality, so we may regard $\varphi^{\nabla}_{\mathrm{M}}$ as an endomorphism of $\varphi^*\mathrm{M}$. One may check that $\varphi^{\nabla}_{\mathrm{M}}$ strictly raises the induced u_0-adic filtration (see [AKN24] Definition 4.54 for the case of $\mathcal{E} = \mathcal{O}$, noting the filtration is shifted by -1). In particular, φ^{∇} is topologically nilpotent, and the convergent series $\sum_{j\geq 0}(\varphi^{\nabla}_{\mathrm{M}})^j$ gives a right inverse to $\mathrm{can} - \varphi^{\nabla}_{\mathrm{M}}$ proving integral surjectivity of α.

In general, we may write the desired reflexive F-gauge as $\mathcal{E}\{r\}$ for some $r \geq 0$, where $\mathcal{E}$ has all Hodge–Tate weights ≥ 0. The corresponding untwisted module $\mathrm{M}\{r\}$ is now equipped with a semilinear map

$$\varphi_{\mathrm{M}\{r\}} : \mathrm{M}\{r\} \longrightarrow \frac{1}{\mathrm{E}(u_0)^r}\mathrm{M}\{r\},$$

so due to the indexing shift this will imply $\varphi^{\nabla}_{\mathrm{M}\{r\}}$ is only defined integrally on $\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\mathrm{M}\{r\} \subseteq \varphi^*\mathrm{M}\{r\}$, and iterates $(\varphi^{\nabla}_{\mathrm{M}\{r\}})^j$ need not be defined on all of $\varphi^*\mathrm{M}\{r\}$ when there is a nontrivial twist. Thus, we cannot run the same argument.

However, a similar argument can be run when working high enough in the F-filtration. The cokernel $\mathrm{coker}(\alpha)$ is derived p-complete (as syntomic cohomology is derived p-complete), and the filtered pieces $\mathrm{F}^i\,\mathrm{coker}(\alpha)$ are derived p-complete as well. Thus, to prove $\mathrm{F}^{i_0}\,\mathrm{coker}(\alpha) = 0$ it suffices to prove $(\mathrm{F}^i\,\mathrm{coker}(\alpha))/p = 0$ for all $i \geq i_0$, i.e. surjectivity of α after restricting to F^i and reducing modulo p.

Since $\mathrm{E}(u_0) \equiv u_0^e \pmod{p}$ (where e is the ramification index), one checks that there is an i_0 so that for $i \geq i_0$ the map $\mathrm{can}-\varphi^{\nabla}_{\mathrm{M}}$ induces a well-defined endomorphism of $(\mathrm{F}^i\varphi^*\mathrm{M}\{r\})/p$ using the fact that on the base $\varphi^*\mathrm{W}(k)[\![u_0]\!]$ corresponding to $\mathcal{O}_{\mathbb{A}}$ the map φ^{∇} scales the -1-shifted F-filtration by a factor of p (again see [AKN24] Definition 4.54, again noting the filtration is shifted by -1). Moreover, for such i we have

$$(\mathrm{F}^i\varphi^*\mathrm{M}\{r\})/p \subseteq (\mathrm{Fil}^{-1}_{\mathrm{Nyg}}\varphi^*\mathrm{M}\{r\})/p,$$

so that passing to $\mathrm{Fil}^{-1}_{\mathrm{Nyg}}$ does not enlarge the source after reduction modulo p. Concretely, writing $i = ej + j'$ with $0 \leq j' < e$, we have $u_0^i \equiv u_0^{j'}\mathrm{E}(u_0)^j \pmod{p}$; for $j \geq r$ we may factor $\mathrm{E}(u_0)^j = \mathrm{E}(u_0)^{j-r}\mathrm{E}(u_0)^r$, and the bounded-denominator condition for the twist implies that multiplying by $\mathrm{E}(u_0)^r$ places us in $\mathrm{Fil}^0_{\mathrm{Nyg}} \subseteq \mathrm{Fil}^{-1}_{\mathrm{Nyg}}$. It follows that, for $i \geq i_0$, we may regard $\mathrm{can} - \varphi^{\nabla}_{\mathrm{M}}$ as an endomorphism of $(\mathrm{F}^i\varphi^*\mathrm{M}\{r\})/p$.

Finally, modulo p the operator $\varphi^{\nabla}_{\mathrm{M}}$ strictly raises the u_0-adic filtration on $(\mathrm{F}^i\varphi^*\mathrm{M}\{r\})/p$ for $i \geq i_0$, hence is topologically nilpotent there. The same convergent power series argument therefore shows that $\mathrm{can} - \varphi_{\mathrm{M}}^{\nabla}$ is surjective onto $(\mathrm{F}^i\varphi^*\mathrm{M}\{r\})/p$ for all $i \geq i_0$, and we conclude that $\mathrm{F}^{i_0}\operatorname{coker}(\alpha) = 0$ integrally (note that this does not show integral surjectivity of $\mathrm{can} - \varphi_{\mathrm{M}}^{\nabla}$, as the cokernel need not be derived p-complete).

It remains to check rationalized surjectivity on the finitely many remaining graded pieces of the F-filtration for indices $i < i_0$. For $i \geq 1$ the constants C_i in the formula $\nabla(u_0 m) = \mathrm{C}_i m + n$ are units after inverting p, so $\mathrm{gr}^i_{\mathrm{F}}\nabla_{\mathrm{M}\{r\}}$ is a rational isogeny on each such graded piece (it acts by C_i on $\mathrm{gr}^i_{\mathrm{F}}$). Since there are only finitely many indices $i < i_0$, it follows that $\operatorname{coker}(\alpha)[1/p] = 0$, as desired. ☐

Proof of Proposition 4.8. We have already shown this for H^0 (by the heart claim in Corollary 4.1) and for H^1 it follows from Proposition 4.3. For $i \geq 2$, the Ext^i groups in $\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)$ are zero. Thus we only need to show vanishing of cohomology in degrees ≥ 2, which was just shown in Proposition 4.12. ☐

Remark 4.13. Let $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ be smooth and quasicompact. There is a hypercohomology Leray spectral sequence

$$\mathrm{E}_1^{i,j} = \mathrm{H}^{2i+j}_{\mathrm{Syn}}(\mathcal{O}_K, \mathrm{R}^{-i}f_*\mathcal{E}) \implies \mathrm{H}^{i+j}_{\mathrm{Syn}}(\mathrm{X}, \mathcal{E}).$$

This is the spectral sequence associated to the filtration on

$$\mathrm{R}\Gamma_{\mathrm{Syn}}(\mathrm{X}, \mathcal{E}) \simeq \mathrm{R}\Gamma_{\mathrm{Syn}}(\operatorname{Spf}\mathcal{O}_K, \mathrm{R}f_*\mathcal{E})$$

by the complexes $\mathrm{RHom}(\mathcal{O}, \tau_{\leq -i}\mathrm{R}f_*\mathcal{E})$ (see Stack Project 015X for more details about the indexing – we choose $\tau_{\leq -i}$ to make a decreasing filtration). We can check this converges using perfectness (hence t-boundedness) of $\mathrm{R}f_*\mathcal{E}$.

This has a practical consequence for smooth proper $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$. Let $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})$.

The differentials on E_1 are

$$d_1^{p,q} : \mathrm{H}^{2p+q}_{\mathrm{Syn}}(\mathcal{O}_K, \mathrm{R}^{-p}f_*\mathcal{E}) \to \mathrm{H}^{2(p+1)+q}_{\mathrm{Syn}}(\mathcal{O}_K, \mathrm{R}^{-p-1}f_*\mathcal{E}).$$

This can never be nontrivial rationally, since H^2 vanishes (as well as all higher H^i). Indeed, $\mathrm{R}^i f_*\mathcal{E}$ is coherent as $\mathrm{R}f_*\mathcal{E}$ is a perfect complex (by Proposition 4.5). Thus after rationalization this spectral sequence degenerates on E_1. Note that this also shows the syntomic period map $\mathrm{H}^i_{\mathrm{Syn}}(\mathrm{X}, \mathcal{E}) \to \mathrm{H}^i_{\text{ét}}(\mathrm{X}, \mathrm{T}_{\text{ét}}(\mathcal{E}))$ is injective rationally in light of Proposition 4.8.

We are now ready to reduce showing the equivalence $\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)) \simeq \mathrm{D}^b(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$ to a cohomology computation. Note that the definition of $\mathrm{RHom}_{\mathrm{MF}_K^{\varphi,\mathrm{wa}}}$ uses the Ind category of $\mathrm{MF}_K^{\varphi,\mathrm{wa}}$ and [Oor64], as the category of weakly admissible filtered isocrystals does not have enough injectives. We can realize the target category $\mathrm{D}^b(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$ as an ∞-category by applying the usual construction for $\mathcal{A} = \mathsf{Ind}(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$ as the stabilization of the animation of the category of injectives and regarding $\mathrm{D}^b(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$ as a full subcategory.

Proposition 4.14. *There is a symmetric monoidal equivalence of categories*

$$\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K))$$

induced by $\mathrm{T}_{\text{ét}}$.

Proof. Once the cohomology calculation has been done, the proof is essentially the same as in [Hau24]. We spell out some details of this argument for the convenience of the reader. The natural equivalence induced by $\mathrm{T}_{\text{ét}}$ is that $\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]) \simeq \mathrm{D}^b(\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta))$ (which is then also equivalent to $\mathrm{D}^b(\mathrm{MF}^{\varphi,\mathrm{wa}}_K)$), so our task is actually to show the natural functor

$$F : \mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]) \to \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$$

is an equivalence for $\mathrm{X} = \operatorname{Spf} \mathcal{O}_K$. Here, we know the functor is well-defined by the same argument as in the proof of Corollary 4.1. This functor can be seen to be essentially surjective once we know full faithfulness, as we have an equivalence on the heart and may construct objects using shifts and fibers from the heart on the target $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ via Lemma 5.4.3 in [Hau24]. We may then construct an object $\mathcal{E} \in \mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p])$ with the desired output $F(\mathcal{E})$ by lifting the attaching maps via full faithfulness and noting F is compatible with shifts and fibers.

Full faithfulness can then be reduced to checking that RHom in both categories agree following the standard reduction to cohomology method of [Hau24] Proposition 5.6.2. The functor F is compatible with duals, tensor products, and the internal RHom, so via the tensor-Hom adjunction we reduce to checking cohomology with coefficients agree (using the isomorphism $\mathrm{RHom}_{\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})}(\mathcal{O}, \mathcal{E}) \simeq \mathrm{R}\Gamma_{\mathrm{Syn}}(\mathcal{O}_K, \mathcal{E})$ to pass to cohomology at the end). Note that existence and compatibility with duals and the internal RHom is slightly tricky for $\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p])$ but can be justified as $\mathrm{D}^b(\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p])$ is equivalent to the rigid symmetric monoidal category $\mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K))$ via the symmetric monoidal functor $\mathrm{T}_{\text{ét}}$. Since F is symmetric monoidal this allows us to deduce F is compatible with duals and RHoms as its source is rigid symmetric monoidal. Finally, compatibility of F with shifts and fibers allows us to further reduce to checking cohomologies with coherent coefficients agree.

Writing out cohomology with coefficients in the heart in each category, since there is a map of complexes we need only check that

$$\mathrm{Ext}^i_{\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)}(\mathbf{Q}_p, \mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]) \simeq \mathrm{H}^i_{\mathrm{Syn}}(\operatorname{Spf} \mathcal{O}_K, \mathcal{E})[1/p]$$

which is shown in Proposition 4.8. □

This result in the case of a point will be useful in proving an analogous result for smooth proper $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$. In light of Example 4.7, it will be necessary to change how the theorem is formulated. It turns out to be more natural to interpret Proposition 4.14 as stating that

$$\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p] \simeq \mathrm{D}^b(\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K)) \simeq \mathrm{D}^b(\mathrm{MF}^{\varphi,\mathrm{wa}}_K)$$

where $\mathrm{MF}^{\varphi,\mathrm{wa}}_K$ is the category of weakly admissible filtered F-isocrystals. This is induced by the abelian category level result of Colmez-Fontaine in [CF00] that the functor $\mathrm{D}_{\mathrm{cris}}$ induces an equivalence $\mathsf{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{G}_K) \simeq \mathrm{MF}^{\varphi,\mathrm{wa}}_K$.

In general, we will need to formulate the appropriate notion of a derived filtered F-isocrystal. This is given by the Beilinson fiber square; namely, we have a commutative diagram of stacks

$$\begin{tikzcd}
\mathrm{X}^{\mathrm{Syn}} & (\mathrm{X}_s)^{\mathrm{Syn}} \arrow[l] \\
(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+} \arrow[u] & (\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}}. \arrow[l] \arrow[u]
\end{tikzcd}$$

where $\mathrm{X}_s := \mathrm{X} \times_{\mathrm{Spf}\,\mathcal{O}_K} \mathrm{Spec}\,k$. There is a nontrivial arrow to define here, namely the map of stacks

$$(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}} \to (\mathrm{X}_s)^{\mathrm{Syn}}.$$

We only need to construct such a map in the case of a point $\mathrm{X} = \mathrm{Spf}\,\mathcal{O}_K$ as this then induces a map for all X. In this case, it is given explicitly by

$$\mathrm{Spf}\,\mathcal{O}_K \to k^{\mathrm{Nyg}} = (\mathrm{Spf}\,\mathrm{W}(k)\langle u, t\rangle/(ut - p))/\mathbf{G}_m \to k^{\mathrm{Syn}}$$

induced by sending $t \to 1$ and $u \to p$. Note that in the unramified case we end up with the inclusion of the $t \neq 0$ locus or $\mathrm{W}(k)^{\mathrm{dR}} \simeq k^{\mathbb{\Delta}} \simeq (k^{\mathrm{Nyg}})_{t\neq 0} \to k^{\mathrm{Syn}}$ as in [Hau24].

What we first show is that after taking cohomology and rationalizing that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathrm{X}^{\mathrm{Syn}}, \mathcal{E})[1/p] \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathrm{X}_s^{\mathrm{Syn}}, \mathcal{E})[1/p] \arrow[d] \\
\mathrm{R}\Gamma((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}, \mathcal{E})[1/p] \arrow[r] & \mathrm{R}\Gamma((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}}, \mathcal{E})[1/p]
\end{tikzcd}$$

becomes Cartesian for $\mathcal{E}$ perfect. This is shown by [Hau24] in the unramified case and in [Dev25] for $\mathbf{Z}_p[\zeta_p]$. Once definitions are unwound we will see this claim is essentially equivalent to Proposition 4.14, similar to the argument in the unramified case in [Hau24].

Corollary 4.15 (Beilinson fiber square for smooth proper X)**.** *Let* $\mathrm{X}/\,\mathrm{Spf}\,\mathcal{O}_K$ *be smooth proper, and denote the residue field of* K *by* k*. Fix a choice of Breuil-Kisin prism* $(\mathrm{W}(k)[\![u_0]\!], \mathrm{E}(u_0))$ *in order to define the maps of stacks as above and let* $\mathcal{E} \in \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})$*. There is a fiber square*

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathrm{X}^{\mathrm{Syn}}, \mathcal{E})[1/p] \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathrm{X}_s^{\mathrm{Syn}}, \mathcal{E})[1/p] \arrow[d] \\
\mathrm{R}\Gamma((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}, \mathcal{E})[1/p] \arrow[r] & \mathrm{R}\Gamma((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}}, \mathcal{E})[1/p]
\end{tikzcd}$$

where we write $\mathrm{X}_s := \mathrm{X} \times_{\mathrm{Spf}\,\mathcal{O}_K} \mathrm{Spec}\,k$ *and abuse notation to again write* $\mathcal{E}$ *for the pullback to the stacks* $\mathrm{X}_s^{\mathrm{Syn}}$, $(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}}$*, and* $(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}$*.*

Proof. By Proposition 4.5, it suffices to prove the claim for $\mathrm{X} = \mathrm{Spf}\,\mathcal{O}_K$ (note that the realizations involved commute with pushforwards, by similar arguments as in Lemma 4.4; see [Hau24]). As already noted, Proposition 4.14 says that the functor

$$\mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p] \to \mathrm{D}^b(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$$

is an equivalence. For $\mathcal{E} \in \mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$, write $\mathrm{D} \in \mathrm{D}^b(\mathrm{MF}_K^{\varphi,\mathrm{wa}})$ for the image under this equivalence.

This is the only nontrivial identification in the square. We can identify $\mathsf{Perf}(k^{\mathrm{Syn}})[1/p]$ with isocrystals $\mathsf{Perf}^{\varphi}(K_0)$ where $K_0 = \mathrm{W}(k)[1/p]$, and as we take the de Rham stacks relative to $\mathcal{O}_K$ we trivially have $\mathsf{Perf}((\mathcal{O}_K/\mathcal{O}_K)^{\mathrm{dR}})[1/p] \simeq \mathsf{Perf}(K)$ and for the filtered de Rham stack we get filtered perfect complexes of K-vector spaces (which we denote MF_K to align with usual notation). Now let D be the associated bounded complex of admissible filtered isocrystals associated to $\mathcal{E} \in \mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[1/p]$, and let $K_0 = \mathrm{W}(k)[1/p]$ where k is the residue field of K. Unwinding definitions we can rewrite the fiber square for the case of $\mathcal{O}_K$ as

$$\begin{tikzcd}
\mathrm{RHom}_{\mathrm{MF}_K^{\varphi,\mathrm{wa}}}(K,\mathrm{D})[1/p] \arrow[r] \arrow[d] & \mathrm{RHom}_{\mathsf{Perf}^{\varphi}(K_0)}(K_0,\mathrm{T}_{\mathsf{cris}}(\mathrm{D})) \arrow[d] \\
\mathrm{RHom}_{\mathrm{MF}_K}(K,\mathrm{T}_{\mathrm{dR},+}(\mathrm{D})) \arrow[r] & \mathrm{RHom}_K(K,\mathrm{T}_{\mathrm{dR}}(\mathrm{D}))
\end{tikzcd}$$

where we obtain the obvious maps. The only one which needs any inspection is the map $\mathsf{Perf}^{\varphi}(K_0) \to \mathsf{Perf}(K)$, which is indeed induced by pullback along the integral map $\mathrm{Spf}\,\mathcal{O}_K \to k^{\mathrm{Syn}}$. We may reduce to D concentrated in degree zero as well by applying Lemma 5.4.3 in [Hau24], at which point the claim follows from the Ext computations in [EK99] (just as in the calculation in [Hau24] Proposition 5.2.4, we use Corollary 2.4.4 in [EK99]) ☐

Remark 4.16. An analogous statement is likely true for all smooth $\mathrm{X}/\mathcal{O}_K$. Using the explicit complex computing integral syntomic cohomology, p-torsion control of the total fiber for reflexive sheaves would be sufficient to deduce this but we do not pursue this here.

Definition 4.17. *Let $\mathrm{X}/\,\mathrm{Spf}\,\mathcal{O}_K$ be a smooth and quasicompact p-adic formal scheme. Define $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ as the isogeny category $\mathcal{C}[1/p]$ of*

$$\mathcal{C} = \mathsf{Perf}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})} \mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}).$$

Individually, all three categories carry a natural t-structure induced by a Noetherian regular flat covering (which is well-defined by the same argument as $\mathrm{X}^{\mathrm{Syn}}$).

The Beilinson fiber square then tells us (using the internal RHom and the tensor-Hom adjunction) in the smooth proper case that we have a fully faithful functor

$$\mathrm{Beil} : \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \to \mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X}).$$

We will next characterize the heart of the t-structure on this category. In what follows the filtration on a filtered F-isocrystal is more general than what is usually allowed, but the essential image of Beil will later be shown to be the subcategory of admissible filtered F-isocrystals and thus land in the usual subcategory with a stronger filtration condition (for example the convention of [GR24] for filtered F-isocrystals).

Definition 4.18. *Let* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *be smooth. The category* $\mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$ *is defined as the isogeny category of* F*-crystals in coherent sheaves on* $(\mathrm{X}_s/\mathrm{W}(k))_{\mathrm{cris}}$.

We set

$$\mathrm{fIsoc}^{\varphi}(\mathrm{X}) := \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k)) \times_{\mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)} \mathsf{Vect}^{\nabla,+}(\mathrm{X}_\eta)$$

where $\mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)$ *is the category of vector bundles* (E,∇) *on* X_η *with flat connection, and* $\mathsf{Vect}^{\nabla,+}(\mathrm{X}_\eta)$ *is the same but with a complete decreasing filtration on* E *so* ∇ *is a filtered map of degree* -1 *with coherent associated graded.*

The functor $\mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k)) \to \mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)$ *is given by* [Ogu84] *Theorem 2.15.*

A filtered F*-isocrystal is admissible if it is of the form* $\mathrm{D}_{\mathrm{cris}}(\mathbb{L})$ *for a crystalline local system* $\mathbb{L}$. *We define* $\mathrm{D}_{\mathrm{cris}}$ *as the composite functor* $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta) \simeq \mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})[1/p] \to \mathrm{fIsoc}^{\varphi}(\mathrm{X})$ *where the second functor is* $\widetilde{\mathrm{D}}_{\mathrm{crys}}$ *from* [GR24] *Proposition 2.38. Let* $\mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ *be the full subcategory consisting of filtered* F*-isocrystals* $\mathcal{E}$ *in perfect complexes where each* $\mathrm{H}^i(\mathcal{E})$ *is admissible.*

This variant of $\mathrm{D}_{\mathrm{cris}}$ is compatible with other definitions which are more classical, for example that in the appendix of [DLMS24]: using Remark 4.7 in [GR24] one need only check the filtered F-isocrystal they associate to a crystalline local system $\mathrm{T}_{\text{ét}}(\mathcal{E})$ is associated in the sense of Faltings (see Definition 2.31 in [GR24]). In particular, the subcategory of admissible objects we use agrees with any other reasonable definition.

The author is not aware of a linear algebraic condition which characterizes admissibility inside of $\mathrm{fIsoc}^{\varphi}(\mathrm{X})$, and it would be interesting to know of one (for example, pointwise weak admissibility at each classical point is likely not sufficient).

The category $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ carries t-structure inherited from the t-structure on each of the categories in the fiber product, constructed analogously from a regular Noetherian flat cover. In the next proposition we verify this is actually well-defined, i.e. that the relevant maps in the fiber product are t-exact on isogeny categories; the category

$$\left(\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Coh}(\mathrm{X}^{\mathrm{dR}}/\mathcal{O}_K)} \mathsf{Coh}(\mathrm{X}^{\mathrm{dR},+}/\mathcal{O}_K)\right)[1/p]$$

is the heart of this t-structure.

Proposition 4.19. *Assume* $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ *is smooth and quasicompact. There is an equivalence of abelian categories*

$$\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \to \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$$

and moreover

$$\left(\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Coh}(\mathrm{X}^{\mathrm{dR}}/\mathcal{O}_K)} \mathsf{Coh}(\mathrm{X}^{\mathrm{dR},+}/\mathcal{O}_K)\right)[1/p] \simeq \mathrm{fIsoc}^{\varphi}(\mathrm{X}).$$

This describes the heart of $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ *with its natural* t*-structure.*

Proof. We start by arguing that $\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \simeq \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$. We have a natural functor

$$\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \to \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$$

given by restriction to X_s^{Δ} (which produces an F-isocrystal for the same reasons as in mixed characteristic). To deduce this is an equivalence, we may use the same argument we used to deduce the analogous claim for $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ and the isogeny category of coherent prismatic F-crystals in mixed characteristic. We may first reduce to when X is smooth affine by Zariski descent. The idea is then that coherent crystals on $(\mathrm{X}_s/\mathrm{W}(k))_{\mathrm{cris}}$ after evaluation on a covering (now taking the prisms $(\tilde{\mathrm{R}}, p)$ where $\tilde{\mathrm{R}}$ is a p-completely smooth $\mathrm{W}(k)$-lift) are vector bundles after inverting p by Corollary 3.10. Up to isogeny, we may then deduce objects on the source and target of $(-)_{\mathrm{X}_s^{\Delta}}$ come from p-torsionfree ones. As remarked in [GL23], the argument in Theorem 3.32 of [GL23] shows these p-torsionfree F-crystals may be fully faithfully lifted via an analogous functor Π_{X_s}. From this we can deduce that the restriction functor $(-)_{\mathrm{X}_s^{\Delta}}$ is an equivalence, by checking the unit of the adjunction $\eta : \mathcal{E} \to \Pi_{\mathrm{X}_s}(\mathcal{E}|_{\mathrm{X}_s^{\Delta}})$ for $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})$ is rationally an equivalence to deduce Π_{X_s} is essentially surjective. Indeed, working locally when X_s is affine we have a flat-local surjection $\mathrm{Spf}(\tilde{\mathrm{R}}\langle u, t\rangle/(ut-p))/\mathbf{G}_m \to (\mathrm{X}_s)^{\mathrm{Syn}}$. We know the kernel and cokernel of η are 0 restricted to X_s^{Δ}, hence when pulled back to the cover they are supported on $\mathrm{V}(t) \subset \mathrm{V}(p)$; it follows that the desired map is rationally an equivalence.

In [Bha22] Remark 2.5.8, explicit descriptions of $\mathsf{Vect}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})$ and $\mathsf{Vect}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})$ are given (and also a general method to describe sheaves, via the identification as a Rees stack). The first category consists of vector bundles with flat connection and nilpotent p-curvature modulo p, and the second category adds a Griffiths transversal decreasing filtration by subbundles. We then have fully faithful functors

$$\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})[1/p] \to \mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)$$

and similarly for the Hodge-filtered de Rham stack with $\mathsf{Vect}^{\nabla,+}$. We observe here that up to p-isogeny $\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})[1/p] \simeq \mathsf{Vect}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})[1/p]$, so we do obtain vector bundles with flat connection after inverting p. For the Hodge-filtered de Rham stack, considering $\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})$ has the same effect on the underlying vector bundle with connection but allows the filtration to be a complete filtration with coherent associated graded; we just ask that ∇ has degree -1. After these identifications we then obtain a fully faithful functor

$$\left(\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Coh}(\mathrm{X}^{\mathrm{dR}}/\mathcal{O}_K)} \mathsf{Coh}(\mathrm{X}^{\mathrm{dR},+}/\mathcal{O}_K)\right)[1/p] \to \mathrm{fIsoc}^{\varphi}(\mathrm{X}),$$

observing the left hand side is well-defined as the pullback from $\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p]$ to $\mathsf{Perf}(\mathrm{X}^{\mathrm{dR}}/\mathcal{O}_K)[1/p]$ induced by the map of stacks in 4.15 indeed lands in the heart (the other map is clear, as it simply forgets the filtration). We may identify this pullback functor, after the first equivalence $\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \to \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$ and unwinding definitions, with passage to the underlying vector bundle with flat connection associated to the isocrystal.

It remains to check essential surjectivity. The essential image of the natural functor

$$\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})[1/p] \to \mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)$$

consists of vector bundles $\mathcal{E}_\eta$ with flat connection where the flat connection arises from a formal model $(\mathcal{E}, \nabla)$ with a flat connection that has nilpotent p-curvature modulo p. By [Lüt90] Lemma 2.2 any coherent sheaf on X_η has a formal model, so $\mathsf{Coh}(\mathrm{X}_\eta) \simeq \mathsf{Coh}(\mathrm{X})[1/p]$. In fact, if $(\mathcal{E}_\eta, \nabla) \in \mathsf{Vect}^{\nabla}(\mathrm{X}_\eta)$ is obtained from $\mathcal{E}' \in \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$, then it has a model locally by [Ogu84] Proposition 2.21 so we also have a formal model for the connection. The sheaf $\mathcal{E}_\eta$ must be locally free, as we may check on an affinoid that the algebraization gives a coherent sheaf with flat connection on a characteristic zero scheme which must be a vector bundle.

The desired claim now follows if we can additionally show there is some formal model for $(\mathcal{E}_\eta, \nabla)$ where ∇ has nilpotent p-curvature modulo p if $(\mathcal{E}_\eta, \nabla)$ comes from some $\mathcal{E}' \in \mathrm{Isoc}^\varphi(\mathrm{X}_s/\mathrm{W}(k))$. This is clear, since the formal model arises from a crystal and must then have a quasi-nilpotent connection, or equivalently nilpotent p-curvature modulo p. The essential image in the filtered case is derived similarly. ☐

Our next goal will be to describe the entire category $\mathsf{Perf}_{\mathrm{fIsoc}^\varphi}(\mathrm{X})$ without reference to a stack.

Definition 4.20. *Assume* $\mathrm{X} = \mathrm{Spf}\,\mathrm{R}$ *is affine over* $\mathrm{Spf}\,\mathcal{O}_K$ *and written as* $\mathrm{X}_0 \widehat{\otimes}_{\mathrm{W}(k)} \mathcal{O}_K$, *where* X_0 *is a model with an étale map*

$$\mathrm{X}_0 \to \mathrm{Spf}\,\mathrm{W}(k)\langle \mathrm{T}_i^{\pm} \rangle$$

so that we have an induced basis of $\widehat{\Omega}^1_{\mathrm{R}}$. *Let* φ *be a global Frobenius lift on* X_η, *available in our local situation via the torus chart.*

A flat connection $\nabla = \sum_i \theta_i \mathrm{dT}_i$ *on a vector bundle* $\mathcal{E}$ *on* X_η *is called convergent if for* $e \in \Gamma(\mathrm{X}_\eta, \mathcal{E})$ *and* $0 \le \eta < 1$ *we have*

$$\lim_{\alpha \to \infty} \left\| \frac{1}{\alpha!} \theta^\alpha(e) \right\| \eta^{|\alpha|} = 0.$$

Here α *is a multi-index.*

Let π *be a uniformizer for* $\mathcal{O}_K$. *We set* $\widehat{\mathrm{D}}_{\mathrm{X}_\eta} := (\widehat{\mathrm{D}}_{\mathrm{X}})[1/\pi]$ *where* $\widehat{\mathrm{D}}_{\mathrm{X}}$ *is the pullback of the sheaf of crystalline differential operators. Let* $\mathsf{Perf}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ *be the category of* $\widehat{\mathrm{D}}_{\mathrm{X}_\eta}$*-modules* $\mathcal{E}$ *whose underlying* $\mathcal{O}_{\mathrm{X}_\eta}$*-module is a perfect complex on* X_η. *For the Hodge-filtered variant, let* $\mathsf{Perf}^+(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ *denote the subcategory of perfect filtered* $\widehat{\mathrm{D}}_{\mathrm{X}_\eta}$*-modules where we endow* $\widehat{\mathrm{D}}_{\mathrm{X}_\eta}$ *with the order filtration. Here, perfectness means the underlying* $\mathcal{O}_{\mathrm{X}_\eta}$*-module is perfect, the associated graded is perfect, and the filtration is complete.*

Finally, we let $\mathsf{Perf}^\varphi(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta})$ *denote the category of* $\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta}$*-modules* $\mathcal{E}$ *with underlying* $\mathcal{O}_{(\mathrm{X}_0)_\eta}$*-module perfect along with an isomorphism* $\varphi^*\mathcal{E} \simeq \mathcal{E}$ *and the condition that each* $\mathrm{H}^i(\mathcal{E})$ *viewed as a vector bundle with connection has a convergent connection.*

We can in fact describe the category $\mathsf{Perf}_{\mathrm{fIsoc}^\varphi}(\mathrm{X})$ explicitly if we are allowed to work Zariski locally and pick a Frobenius lift. The same argument as Lemma 2.9 in [GR24] justifies that we can locally land in the situation of Definition 4.20.

Proposition 4.21. *Assume* X_0 *is affine and étale over* $\mathrm{Spf}\,\mathrm{W}(k)\langle \mathrm{T}_i^{\pm} \rangle$, *and choose a global Frobenius lift* φ *on* X_0. *Then*

$$\mathsf{Perf}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \simeq \mathsf{Perf}^\varphi(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta}).$$

Let $\mathrm{X} = \mathrm{X}_0 \widehat{\otimes}_{\mathrm{W}(k)} \mathcal{O}_K$*. Then there is also an equivalence*

$$\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X}) \simeq \mathsf{Perf}^{\varphi}(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta}) \times_{\mathsf{Perf}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})} \mathsf{Perf}^{+}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta}).$$

Proof. We consider the claim $\mathsf{Perf}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \simeq \mathsf{Perf}^{\varphi}(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta})$ first. The crystalline realization defines a functor

$$\mathsf{Perf}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \to \mathsf{Perf}(\mathrm{X}_{s,\mathrm{cris}}, \mathcal{O}_{\mathrm{cris}})[1/p]^{\varphi=1}.$$

We will deduce this functor is fully faithful using the fiber sequence

$$\mathrm{R}\Gamma(\mathrm{X}_s^{\mathrm{Syn}}, \mathcal{E}) \longrightarrow \mathrm{Fil}^0_{\mathrm{Nyg}} \mathrm{R}\Gamma(\mathrm{X}_s^{\Delta}, \mathcal{E}) \xrightarrow{\varphi_{\mathcal{E}}-1} \mathrm{R}\Gamma(\mathrm{X}_s^{\Delta}, \mathcal{E}).$$

In our situation X_s is equipped with a lift $\mathrm{Spf}\,\mathrm{R}$, which also has a global Frobenius lift due to the framing we are working in. This means that crystalline cohomology can be computed via the completed de Rham complex $\widehat{\Omega}_{\mathrm{X}}(\mathcal{E})$ of $\mathcal{E}$, and the Nygaard filtration on $\mathrm{R}\Gamma(\mathrm{X}_s^{\mathrm{cris}}, \mathcal{E}) \simeq \varphi^* \mathrm{R}\Gamma(\mathrm{X}_s^{\Delta}, \mathcal{E})$ will have kth filtered piece given by

$$p^k \mathcal{E} \otimes \Omega^0_{\mathrm{R}} \to p^{k-1} \mathcal{E} \otimes \widehat{\Omega}^1_{\mathrm{R}} \to \cdots \to p\mathcal{E} \otimes \widehat{\Omega}^{k-1}_{\mathrm{R}} \to \mathcal{E} \otimes \widehat{\Omega}^k_{\mathrm{R}} \to \mathcal{E} \otimes \widehat{\Omega}^{k+1}_{\mathrm{R}} \to \cdots$$

where $\mathcal{E}$ is the corresponding object on $\mathrm{X}_0^{\mathrm{dR}} \simeq \mathrm{X}_s^{\mathrm{cris}}$. This description works when $\mathcal{E}$ is in the heart and p-torsionfree.

It follows that inverting p we get the fiber sequence

$$\mathrm{R}\Gamma(\mathrm{X}_s^{\mathrm{Syn}}, \mathcal{E})[1/p] \longrightarrow \mathrm{R}\Gamma(\mathrm{X}_s^{\Delta}, \mathcal{E})[1/p] \xrightarrow{\varphi_{\mathcal{E}}-1} \mathrm{R}\Gamma(\mathrm{X}_s^{\Delta}, \mathcal{E})[1/p],$$

for $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})$. Here we use that crystalline cohomology is recovered as the Frobenius pullback of prismatic cohomology in positive characteristic so that

$$\mathrm{R}\Gamma_{\mathrm{cris}}(\mathrm{X}_s, \mathcal{E})[1/p]^{\varphi_{\mathcal{E}}=1} \simeq \mathrm{R}\Gamma_{\Delta}(\mathrm{X}_s, \mathcal{E})[1/p]^{\varphi_{\mathcal{E}}=1}.$$

Then we use the description of the Nygaard filtration; as mentioned in the proof of Proposition 4.19 we may assume $\mathcal{E}$ is p-torsionfree as this is always true up to isogeny. The same argument as Proposition 4.14 allows us to deduce full faithfulness from this result, as well as essential surjectivity once we know full faithfulness as we've shown the claim on the heart in Proposition 4.19.

Now we can show

$$\mathsf{Perf}(\mathrm{X}_{s,\mathrm{cris}}, \mathcal{O}_{\mathrm{cris}})[1/p]^{\varphi=1} \to \mathsf{Perf}^{\varphi}(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta})$$

is an equivalence; we can compute crystalline cohomology as the completed de Rham cohomology of $\mathrm{X}_0/\mathrm{W}(k)$. This then rationalizes to de Rham cohomology on $(\mathrm{X}_0)_\eta$, so we see that the functor is fully faithful by the same "reduction to cohomology" argument used in Proposition 4.14. Essential surjectivity on the heart follows from the fact that $\mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$ is equivalent to the category of vector bundles $\mathcal{E}$ on $(\mathrm{X}_0)_\eta$ with a flat convergent connection ∇ and an isomorphism $\varphi^*\mathcal{E} \simeq \mathcal{E}$ by [Ogu84] Proposition 2.18. Now that we know full faithfulness, since all t-structures we work with are bounded and nondegenerate we see essential surjectivity follows from compatibility of the functor with shifts and fibers, using [Hau24] Lemma 5.4.3 and the equivalence on the heart (similar to Proposition 4.14). Finally, the functor $\mathsf{Perf}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \to \mathsf{Perf}(\mathrm{X}_{s,\mathrm{cris}}, \mathcal{O}_{\mathrm{cris}})[1/p]^{\varphi=1}$ is an equivalence on the heart by the equivalence $\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \simeq \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$ of Proposition 4.19. This again extends to essential surjectivity by the same argument.

Now we turn to the second claim. To analyze $\mathsf{Perf}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ and $\mathsf{Perf}^+(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ we will not actually need the local form of X (only that it is smooth). We have fully faithful functors $\mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})[1/p] \to \mathsf{Perf}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ as well as the filtered variant

$$\mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})[1/p] \to \mathsf{Perf}^+(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$$

with essential images consisting of objects $(\mathcal{E}_\eta, \nabla)$ admitting an integral model on X where each cohomology sheaf has a connection with nilpotent p-curvature modulo p. To produce these functors, one way (among many) is to use a local presentation of the stack X. As X is smooth, Zariski locally we may assume X is étale over the formal scheme $\mathbf{A}^n_{\mathcal{O}_K} = \mathrm{Spf}\,\mathcal{O}_K\langle \mathrm{T}_1, \dots, \mathrm{T}_n\rangle$, $(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+} \simeq (\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m)/\mathrm{V}(\mathcal{O}(-1))^{\#,n}$ (the $\mathrm{V}(\mathcal{O}(-1))^{\#,n}$ torsor structure given by pulling back the case of $\mathbf{A}^n_{\mathcal{O}_K}$ along the étale map). Using this description of the stack and filtered Koszul duality, quasicoherent sheaves on the stack can be interpreted as filtered D-modules for the algebra of crystalline differential operators with the order filtration. We may use the proof of [BL22b] Lemma 6.7 to translate $\Gamma^*(\mathrm{E})$ coactions to continuous $\mathrm{Sym}^*(\mathrm{E})$ actions. In particular we can apply this to $\mathsf{QCoh}([(\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m)/\mathrm{V}(\mathrm{E})^{\#}])$ for $\mathrm{V}(\mathrm{E})^{\#}$ acting on $\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m$ to see this category is equivalent to $\mathsf{Mod}_{\mathrm{Sym}^*(\mathrm{E}^\vee)}(\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m)$, the category of quasicoherent sheaves on $\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m$ equipped with a continuous action of $\widehat{\mathrm{Sym}^*}$ (completed with respect to the $\mathrm{Sym}^{\geq n}$ filtration) compatible with the action of $\widehat{\mathrm{Sym}^*(\mathrm{E}^\vee)}$ on $\mathcal{O}_{\mathrm{X} \times \mathbf{A}^1/\mathbf{G}_m}$. Rationalizing, we see sheaves on this stack are a full subcategory of filtered $\widehat{\mathrm{D}}_{\mathrm{X}_\eta}$-modules with the order filtration; we can check perfectness on the open point $\mathbf{A}^1/\mathbf{G}_m$ and the formal completion on the closed point, resulting in the definition of $\mathsf{Perf}^+(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$. Note that the filtration being complete with perfect associated graded encodes the second condition. Knowing the first equivalence, we deduce full faithfulness and may again test essential surjectivity in the second claim $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X}) \simeq \mathsf{Perf}^{\varphi}(\widehat{\mathrm{D}}_{(\mathrm{X}_0)_\eta}) \times_{\mathsf{Perf}(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})} \mathsf{Perf}^+(\widehat{\mathrm{D}}_{\mathrm{X}_\eta})$ by reducing to the equivalence on the heart shown in Proposition 4.19. □

On the heart, the functor Beil then identifies with the functor $\mathrm{D}_{\mathrm{cris}} : \mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta) \to \mathrm{fIsoc}^{\varphi}(\mathrm{X})$. This will allow us to characterize the essential image as objects whose cohomology sheaves are admissible.

Proposition 4.22. *The diagram*

$$\begin{tikzcd}
\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p] \arrow[r, "\mathrm{Beil}"] \arrow[d, "\mathrm{T}_{\text{ét}}"] & \left(\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})} \mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})\right)[1/p] \arrow[d, "\sim"] \\
\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta) \arrow[r, "\mathrm{D}_{\mathrm{cris}}"] & \mathrm{fIsoc}^{\varphi}(\mathrm{X})
\end{tikzcd}$$

commutes.[5] *Recall that the right vertical arrow is an equivalence by Proposition 4.19.*

Proof. It suffices to check the claim locally, so we can even assume that $\mathrm{X} = \mathrm{X}_0 \widehat{\otimes}_{\mathrm{W}(k)} \mathcal{O}_K$ where X_0 is étale over $\mathrm{W}(k)\langle \mathrm{T}_1^{\pm}, \ldots, \mathrm{T}_n^{\pm}\rangle$ (this reduction follows from Lemma 2.9 in [GR24]).

It suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p] \arrow[r, "\mathrm{Beil}"] \arrow[d, "(-)|_{\mathrm{X}^{\Delta}}"] & \left(\mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}}) \times_{\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})} \mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})\right)[1/p] \arrow[d, "\sim"] \\
\mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})[1/p] \arrow[r, "\tilde{\mathrm{T}}_{\mathrm{cris}}"] & \mathrm{fIsoc}^{\varphi}(\mathrm{X})
\end{tikzcd}$$

Here, $\tilde{\mathrm{T}}_{\mathrm{cris}}$ is the fully faithful functor to filtered F-isocrystals constructed in Corollary 4.6 of [GR24]. The commutativity of the top square is easy to see once we recall $\tilde{\mathrm{T}}_{\mathrm{cris}}$ is given by taking the restriction functor

$$\mathrm{T}_{\mathrm{cris}} : \mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})[1/p] \to \mathsf{Coh}^{\varphi}(\mathrm{X}_{s,\Delta}, \mathcal{O}_{\Delta})[1/p] \simeq \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))$$

and using Proposition 2.38 of [GR24] to equip this with a filtration to obtain $\tilde{\mathrm{T}}_{\mathrm{cris}}$. We have already shown in Proposition 4.19 that $\mathrm{T}_{\mathrm{cris}}$ fits in the diagram

$$\begin{tikzcd}
\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p] \arrow[r, "(-)|_{\mathrm{X}_s^{\mathrm{Syn}}}"] \arrow[d, "(-)|_{\mathrm{X}^{\Delta}}"] & \mathsf{Coh}(\mathrm{X}_s^{\mathrm{Syn}})[1/p] \arrow[d, "(-)|_{\mathrm{X}_s^{\Delta}}"] \\
\mathsf{Coh}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})[1/p] \arrow[r, "\mathrm{T}_{\mathrm{cris}}"] & \mathrm{Isoc}^{\varphi}(\mathrm{X}_s/\mathrm{W}(k))
\end{tikzcd}$$

so we need only verify that we obtain the same filtration from both constructions. By Proposition 2.38 of [GR24], what we need to show is that given $\mathcal{E} \in \mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ we have a canonical identification

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]) \simeq \mathcal{E}|_{(\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}}[1/p]$$

via the fully faithful functor $\mathsf{Coh}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+}) \to \mathsf{Vect}^{\nabla,+}(\mathrm{X}_\eta)$ allowing us to regard both sides as filtered vector bundles on X_η with a flat Griffiths transversal connection. The argument in [Hau24] Remark 7.2.9 shows this is the case (the same reasoning works relative to $\mathcal{O}_K$ when $\mathrm{X}/\operatorname{Spf}\mathcal{O}_K$ is smooth). □

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[5] Recall we define $\mathrm{D}_{\mathrm{cris}}$ using [GR24] Corollary 4.6, using the equivalence $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Q}_p}(\mathrm{X}_\eta) \simeq \mathsf{Vect}^{\varphi,\mathrm{an}}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta})[1/p]$ and then applying their fully faithful functor to filtered F-isocrystals (which we call $\tilde{\mathrm{T}}_{\mathrm{cris}}$ for consistency here).

Theorem 4.23. *Assume* X *is smooth proper over* $\mathrm{Spf}\ \mathcal{O}_K$*. Then* Beil *induces an equivalence*

$$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}})[1/p] \simeq \mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X}).$$

This equivalence is t*-exact and symmetric monoidal.*

Proof. To deduce the claimed t-exactness, recall we defined the t-structure on the target to come from the individual t-structures on $\mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR},+})$, $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}_s)$ and $\mathsf{Perf}((\mathrm{X}/\mathcal{O}_K)^{\mathrm{dR}})$; the pullbacks to each of these are t-exact rationally. Of these the only non-obvious one is the pullback to $\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}_s)$, which can be seen by showing the derived pullback of objects in $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}})[1/p]$ lands in $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}_s)[1/p]$; this follows since we have already seen that up to isogeny we may choose a reflexive representative. By descent we may work Zariski locally to assume $\mathrm{X} = \mathrm{Spf}\ \mathrm{R}$, where we have a cover $\mathrm{Rees}_{\mathrm{E}(u_0)^{\bullet}}\mathrm{R}_0[\![u_0]\!]$ of $\mathrm{X}^{\mathrm{Syn}}$ induced by a corresponding Breuil-Kisin prism with a compatible cover $\mathrm{Rees}_{p^{\bullet}}\mathrm{R}_0 \to \mathrm{X}^{\mathrm{Syn}}_s$ given by the $u_0 = 0$ locus. By this, we mean there is a commutative square

$$\begin{tikzcd}
\mathrm{Rees}_{p^{\bullet}}\mathrm{R}_0 \arrow[r] \arrow[d] & \mathrm{Rees}_{\mathrm{E}(u_0)^{\bullet}}\mathrm{R}_0[\![u_0]\!] \arrow[d] \\
\mathrm{X}^{\mathrm{Syn}}_s \arrow[r] & \mathrm{X}^{\mathrm{Syn}}.
\end{tikzcd}$$

The existence of the reflexive (hence u_0-torsionfree) representative shows the derived pullback to the cover $\mathrm{Rees}_{p^{\bullet}}\mathrm{R}_0$ of $\mathrm{X}^{\mathrm{Syn}}_s$ is coherent, hence the pullback lands in $\mathsf{Coh}(\mathrm{X}^{\mathrm{Syn}}_s)[1/p]$.

By Corollary 4.15, we know Beil is fully faithful so we need only show that the essential image is $\mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$. Each $\mathrm{H}^i(\mathrm{Beil}(\mathcal{E}))$ is an admissible filtered F-isocrystal by Proposition 4.22 and using t-exactness of Beil, we see the essential image of Beil is a full subcategory of $\mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$.

The functor Beil is symmetric monoidal essentially by construction, as the functor on each component of $\mathsf{Perf}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ is induced by a pullback. To show every object of $\mathsf{Perf}^{\mathrm{adm}}_{\mathrm{fIsoc}^{\varphi}}(\mathrm{X})$ is in the essential image, since we already know Beil is fully faithful in the derived sense, t-exact, and an isomorphism on the heart, Lemma 5.4.3 in [Hau24] can be used to deduce essential surjectivity as the t-structures are bounded and nondegenerate. ☐

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