THE PRO-ETALE TOPOLOGY FOR RIGID SPACES

1. The pro-etale topology

Let X be a rigid analytic space over a non-Archimedean field K. Recall that a rigid analytic space can be viewed as an adic space which is covered by affinoids which are topologically of finite type.

Definition 1.1. A morphism $\mathrm{X} \to \mathrm{Y}$ of rigid analytic spaces is called étale if it is locally of finite presentation and for any Huber pair $(\mathrm{A}, \mathrm{A}^+)$ and any ideal $\mathrm{I} \leq \mathrm{A}$ with $\mathrm{I}^2 = 0$ we have a diagram

$$\begin{array}{ccc} \mathrm{X} & \longleftarrow & \mathrm{Spa}((\mathrm{A}, \mathrm{A}^+)/\mathrm{I}) \\ \downarrow & & \downarrow \\ \mathrm{Y} & \longleftarrow & \mathrm{Spa}(\mathrm{A}, \mathrm{A}^+) \end{array}$$

and there exists a unique map $\mathrm{Spa}((\mathrm{A}, \mathrm{A}^+) \to \mathrm{X}$ making the diagram commute. Here, $(\mathrm{A}, \mathrm{A}^+)/\mathrm{I}$ is the Huber pair $(\mathrm{A}/\mathrm{I}, \mathrm{A}^+/(\mathrm{A}^+ \cap \mathrm{I})^{\mathrm{int}})$.

This is meant to mimic the definition of formally étale. Note that you can make the same definition for general adic spaces, but it will actually not agree with the étale site for a perfectoid space.

If you look at the étale topology for a general adic space (e.g. a perfectoid space), one defines finite étale morphisms on affinoids

$$(\mathrm{A}, \mathrm{A}^+) \to (\mathrm{B}, \mathrm{B}^+)$$

to be a morphism where $\mathrm{A} \to \mathrm{B}$ is finite étale with the induced topology and B^+ is the integral closure of A^+.

A general étale morphism is a map $f : \mathrm{X} \to \mathrm{Y}$ such that for every point $x \in \mathrm{X}$ there's open neighborhoods U, V of $x, f(x)$ and a factorization

$$\begin{array}{ccc} \mathrm{U} & \xrightarrow{\iota} & \mathrm{W} \\ & \searrow^{f|_{\mathrm{U}}} & \downarrow c \\ & & \mathrm{V} \end{array}$$

where c is finite étale and ι is an open immersion. This agrees with the above definition for rigid spaces. Observe the corresponding factorization in schemes only has c finite, and not necessarily unramified.

> REMARK 1.2. The analogous factorization *does* hold for complex manifolds. You can use small open balls.

With this, one defines the pro-étale site as follows.

> DEFINITION 1.3. Let $X_{\text{proét}}$ be the category whose objects consist of formal cofiltered limits
> $$U = \varprojlim U_i$$
> where the U_i are rigid-analytic spaces étale over X. We ask that the transition maps $U_i \to U_j$ commute with the maps to X, and are *finite* étale and surjective for $i \gg j$. The morphisms are induced by viewing this as a subcategory of $\mathrm{Pro}(X_{\text{ét}})$.

Now we wish to put a Grothendieck topology on this category. We assign U as a topological space
$$|U| = \varprojlim_i |U_i|.$$
A covering is roughly then a family $f_\alpha : U_\alpha \to U$ of pro-étale morphisms such that the images of $f_\alpha(|U_\alpha|)$ cover $|U|$. To be entirely correct, we need some set theoretic conditions but we will ignore these.

Observe that since an étale cover is a pro-étale cover, there is a natural morphism of sites
$$\nu : X_{\text{proét}} \to X_{\text{ét}}.$$
We define
$$\mathcal{O}^+ = \nu^* \mathcal{O}^+_{X,\text{ét}}.$$
We will typically study its p-completion $\widehat{\mathcal{O}}^+$.

Perfectoid spaces

Perfectoid spaces are objects in the larger category of adic spaces that contains rigid analytic spaces.

Let K be a perfectoid field. This means the following:

- The field K is nonarchimedean (of residue characteristic p) and not discretely valued.
- We have $|p| < 1$.
- The p-power map $\mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.

Let me attempt to explain these conditions via example. The first real example of a perfectoid field is $K = \widehat{\mathbf{Q}_p(p^{1/p^\infty})}$. One has the following striking theorem:

> THEOREM 1.4 (Fontaine-Wintenberger, rephrased). There is an isomorphism
> $$\mathrm{G}_{\widehat{\mathbf{Q}_p(p^{1/p^\infty})}} \simeq \mathrm{G}_{\widehat{\mathbf{F}_p(t^{1/p^\infty})}}.$$
> Note that both of these are perfectoid fields.

Let us see how this arises, without actually giving a proof. The idea is that perfectoid fields give rise to tilts $K^\flat$ which are characteristic p perfectoid field. Formally, these have

$$\mathcal{O}_{K^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_K \quad K^\flat = \varprojlim_{x \mapsto x^p} K.$$

Componentwise multiplications make this a multiplicative monoid, but then deducing the ring structure requires the perfectoid field axioms. There is a map

$$\# : K^\flat \to K$$

sending x to its zeroth component. The previous example is of a perfectoid field and its tilt. One can show finite extensions are perfectoid, and this tilting constructing induces a bijection between finite extensions.

Scholze showed that this construction has a geometric generalization in adic spaces.

> DEFINITION 1.5. Let K be a perfectoid field containing $\mathbf{Q}_p$. A topological K-algebra R is perfectoid if the following conditions hold:
>
> - R is uniform, so R° is bounded in R.
> - R° is p-adically complete.
> - There exists a pseudo-uniformizer $\varpi \in \mathrm{R}^\circ$ such that $\varpi^p | p$ and the p-power map
> $$\mathrm{R}^\circ/\varpi \to \mathrm{R}^\circ/\varpi^p$$
> is an isomorphism.

We can replace the last condition with $\mathrm{R}^\circ/p \to \mathrm{R}^\circ/p$ is a surjection, or that R°/p is a semiperfect $\mathbf{F}_p$-algebra, in the case that ϖ exists. This is always the case if the algebra is over a perfectoid field.

We call a Huber pair $(\mathrm{R}, \mathrm{R}^+)$ perfectoid affinoid if R is perfectoid.

> THEOREM 1.6 (Scholze). $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ is an adic space.

This is actually nontrivial to show, since you need to verify that the Huber pair $(\mathrm{R}, \mathrm{R}^+)$ is actually sheafy. The way this is done in the Scholze's original paper it to prove that $(\mathrm{R}, \mathrm{R}^+)$ is stably uniform, by showing the stronger claim that rational opens are perfectoid. Stable uniformity implies that $(\mathrm{R}, \mathrm{R}^+)$ is sheafy.

The reason one typically cares about perfectoid spaces is the following theorem, which is similar to what we saw before in the case of a field (but now much more difficult to prove).

Theorem 1.7 (Scholze). Let $\mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$. Setting

$$\mathrm{R}^\flat = \varprojlim_{x \mapsto x^p} \mathrm{R}$$

and similarly for $\mathrm{R}^{+\flat}$, we obtain a tilted Huber pair $(\mathrm{R}^\flat, \mathrm{R}^{+\flat})$. Then we have

$$\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)_{\text{ét}} \simeq (\mathrm{Spa}\, \mathrm{R}^\flat, \mathrm{R}^{\flat +})_{\text{ét}},$$

and moreover tilting induces a homeomorphism of the adic spectra.

The importance of this is that if one can reduce to studying perfectoid objects in some argument, then via the tilting equivalence we can work in characteristic p.

In the context of $\mathrm{X}_{\text{proét}}$, we can observe that the actual U_i in a formal cofiltered limit are rigid analytic and cannot be perfectoid. However, we can interpret a formal filtered colimit as a perfectoid space.

Definition 1.8. An affinoid perfectoid object in $\mathrm{X}_{\text{proét}}$ is a formal cofiltered limit $\varprojlim \mathrm{U}_i$ where $\mathrm{U} = \mathrm{Spa}(\mathrm{R}_i, \mathrm{R}_i^+)$ such that

$$(\mathrm{R}, \mathrm{R}^+) := \Big((\widehat{\varinjlim \mathrm{R}_i^+})[1/p], (\widehat{\varinjlim \mathrm{R}_i^+}) \Big)$$

is an affinoid perfectoid Huber pair.

We can now call an object U in $\mathrm{X}_{\text{proét}}$ *perfectoid* if it has an open cover by affinoid perfectoid $\mathrm{V} \subset \mathrm{U}$ (here we use that quasicompact open subsets again give rise to objects in $\mathrm{X}_{\text{proét}}$).

Theorem 1.9 (Scholze, Colmez). Let X be a rigid analytic space over $\mathrm{Spa}(K, K^+)$ where K is a field containing $\mathbf{Q}_p$. Affinoid perfectoid objects in $\mathrm{X}_{\text{proét}}$ form a basis for the topology.

Sketch. It suffices to do the case where $\mathrm{X} = \mathrm{Spa}(\mathrm{A}, \mathrm{A}^+)$. We want to produce a sequence of finite étale extensions A_i/A which assemble into a perfectoid affinoid object in $\mathrm{X}_{\text{proét}}$.

Giving such a uniform construction suffices, since actually if $\tilde{X} = \varprojlim_i \tilde{X}_i$ is perfectoid in $X_{\text{proét}}$ then for any $V \to \tilde{X}$ pro-étale we know V is perfectoid. We can assume V is an inverse system of surjective finite étale morphisms with $V_0 = U$. Indeed, we can factor always as $V \to V_0 \to U$ where the last morphism is étale. But then it is locally built out of rational subsets and finite étale covers (which preserve being perfectoid), so without loss of generality we may take $V_0 = U$.

To see this claim that V is perfectoid, we first note that in $V = \varprojlim_j V_j$, each V_j must arise as a pullback of $V_{ij} \to U_i$ which is finite étale, i.e. $V_j = V_{ij} \times_{U_i} U$. Taking the limit of these over i, we obtain a perfectoid space. Thus, V is perfectoid, since perfectoid spaces are closed under completed cofiltered limits.

Hence, the covering of arbitrary $U \in X_{\text{proét}}$ is given by the fiber product $U \times_X \tilde{X}$, as this is pro-étale over U.

Some minor reductions allow us to assume A has no idempotents, and further that X lives over an algebraically closed field. At this point, the actual construction is due to Colmez. The idea is to take the $\mathbf{F}_p$ vector space

$$V = (1 + A^{\circ\circ})/(1 + A^{\circ\circ})^p$$

where $A^{\circ\circ}$ are the topologically nilpotent elements. Set $A_1 := \varinjlim_S A_S$ over finite dimensional subspaces $S \subset V$. Given the preimage $\tilde{S} \subset (1 + A^{\circ\circ})$, we define

$$A_S := A \otimes_{\mathbf{Z}[\tilde{S}]} \mathbf{Z}[\tilde{S}]$$

with the morphism being induced by $[s] \mapsto [s^p]$. We can think of this as formally adjoining pth roots of lifts of a basis of S. As each A_S is finite étale (here we use characteristic zero), they are again uniform Tate with A_S° being the integral closure of A° in A_S. Then A_1 is uniform.

Colmez calls the ith iterate of this construction A_i. These produce finite étale extensions, and the completed colimit of these rings A_∞ is perfectoid. Observe that

$$(1 + A_\infty^{\circ\circ})/(1 + A_\infty^{\circ\circ})^p = \varinjlim_n (1 + A_n^{\circ\circ})/(1 + A_n^{\circ\circ})^p.$$

As transition maps here are all zero, it follows that elements in $(1 + A_\infty^{\circ\circ})$ are now all p-power. Given a pseudouniformizer ϖ then setting $x_n = \sqrt[p^n]{1 + \varpi}$ we have $x_n - 1$ as a psuedo-uniformizer with $(x_n - 1)^p | p$ for $n \gg 0$.

We can also see that the p-power map

$$A_\infty^\circ/(x_n - 1) \to A_\infty^\circ/(x_n - 1)^p$$

is surjective. □

REMARK 1.10. We can characterize this construction as exactly the perfectoid Tate-Huber pairs $(\mathrm{R}, \mathrm{R}^+)$ such that

$$\log : 1 + \mathrm{R}^{\circ\circ} \to \mathrm{R}$$

is a surjection.

REMARK 1.11. If X is smooth over a perfectoid field (say $K = \mathbf{C}_p$), it locally admits an étale map to $\mathbb{T}^n = \mathrm{Spa}(K\langle \mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}\rangle, K^+\langle \mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}\rangle)$. As an adic space, for example $\mathbb{T}^1$ has classical points that look like $\mathcal{O}_{\mathbf{C}_p}^\times$ so it behaves like a circle (but still has thickness, because these still contain open balls).

We can explicitly make the cover in this case, via the perfectoid torus $\tilde{\mathbb{T}}^n$! This is explicitly given as an affinoid perfectoid object in the pro-étale site by the formal cofiltered limit of $\mathrm{Spa}(K\langle \mathrm{T}_1^{\pm 1/p^m}, \dots, \mathrm{T}_n^{\pm 1/p^m}\rangle, K^+\langle \mathrm{T}_1^{\pm 1/p^m}, \dots, \mathrm{T}_n^{\pm 1/p^m}\rangle)$ (so when we take the completed colimit of the rings, we get a perfectoid space; it is necessary that K be perfectoid for this to happen).

More precisely, up to replacing X by an affinoid cover we have an étale map

$$\mathrm{X} \to \mathbb{T}^n.$$

Then $\mathrm{X} \times_{\mathbb{T}^n} \widetilde{\mathbb{T}^n}$ is the desired space pro-étale over X.

Condensed pro-étale cohomology

Given a rigid analytic space X, we will be interested in understanding how to compute $\mathrm{H}^\bullet(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+)$. Knowing that the pro-etale topology has a basis given by perfectoids, we might consider a pro-etale affinoid perfectoid cover $\mathrm{U}_i \to \mathrm{X}$. Then we want to appeal to the following fact about cohomology of $\widehat{\mathcal{O}}^+$ on perfectoid affinoids:

THEOREM 1.12. For $i \geq 1$, the cohomology groups $\mathrm{H}^i(\mathrm{U}_{\text{proét}}, \widehat{\mathcal{O}}^+)$ are almost zero (so they are killed by ϖ_n for all n, where ϖ_n is an element so $\varpi_n^{p^n} = \varpi \cdot u$ for $u \in (\mathrm{R}^\circ)$).

This means the cover allows us to almost compute $\mathrm{H}^\bullet(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+)$. We have an almost isomorphism

$$\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+) \simeq^a \widehat{\mathcal{O}}^+(\mathrm{X}) \to \prod_i \widehat{\mathcal{O}}^+(\mathrm{U}_i) \to \prod_i \widehat{\mathcal{O}}^+(\mathrm{U}_i \times_{\mathrm{X}} \mathrm{U}_j) \to \dots$$

as by descent we would usually put $\mathrm{R}\Gamma_{\text{proét}}$ (where we have now replaced by H^0, noting that everything is affine).

For future applications, it will be important to understand pro-étale cohomology of the associated sheaf of condensed groups $\widehat{\mathcal{O}}^+_{\mathrm{cond}}$. The construction of this condensed sheaf is very simple: to U, we just take the associated condensed abelian group $\underline{\widehat{\mathcal{O}}^+(\mathrm{U})}$, where here we use the topological abelian group structure. Note that this is a sheaf because the assignment $\mathrm{A} \mapsto \underline{\mathrm{A}}$ preserves limits (in particular equalizers).

We first note that there are two ways we could get an output for cohomology which lives in $\mathrm{D}(\mathrm{Cond}(\mathsf{Ab}))$: one way is to take the morphism

$$\lambda : \mathrm{X}_{\text{proét}} \to *_{\text{proét}}$$

and define $\mathrm{R}\Gamma_{\mathrm{cond}}(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+) := \mathrm{R}\lambda_* \widehat{\mathcal{O}}^+$. Then since it lands on the pro-etale site of point and is valued in groups, we get a complex of condensed abelian groups.

LEMMA 1.13. These constructions agree:

$$\mathrm{R}\Gamma_{\mathrm{cond}}(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+) \simeq \mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}}).$$

Proof. This is relatively straightforward. Since we have a basis for the topology given by affinoid perfectoids, it suffices to verify the claim there. In particular, we just need to check that

$$\lambda_* \widehat{\mathcal{O}}^+ \simeq \widehat{\mathcal{O}}^+_{\mathrm{cond}}(\mathrm{U}).$$

On an affinoid $\mathrm{U} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$, this actually only depends on R^+ being p-adically complete (which is true for perfectoid spaces).

Indeed, testing equality of these sheaves amounts to verifying that for a profinite set S we have

$$\mathrm{H}^0(\mathrm{U} \times S, \widehat{\mathcal{O}}^+) = \widehat{\mathcal{O}}^+_{\mathrm{cond}}(\mathrm{U})(S).$$

The right hand side is the same as $C_{\mathrm{cts}}(S, \mathrm{R}^+)$. But then the other side is naturally

$$\mathrm{H}^0(\mathrm{U} \times S, \widehat{\mathcal{O}}^+) \simeq \left(\varinjlim_i C_{\mathrm{cts}}(S_i, \mathrm{R}^+) \right)^{\wedge}_{(p)}.$$

Now we can appeal to p-adic completeness of R^+ to see this equals $C_{\mathrm{cts}}(S, \mathrm{R}^+)$. □

As we saw in the introductory talk, it will be important to use the condensed machinery to make sense of statements like

$$\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}}) \simeq \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{h\mathrm{G}}$$

where Y is a pro-étale G-torsor for some profinite group G. This will become useful for when we want to move between the Lubin-Tate and Drinfeld towers. The proof of this is actually extremely straightforward: a crucial thing is that the map

$$\mathrm{G} \times \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$$

sending $(g,y) \mapsto (y, gy)$ is an isomorphism. Then computing $\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})$ via descent gives it as the totalization of

$$\mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}) \longrightarrow \mathrm{R}\Gamma((\mathrm{G}\times\mathrm{Y})_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}) \rightrightarrows \dots$$

This is the same as the totalization of the cosimplicial object $\mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}[\mathrm{G}^\bullet], \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}))$. To see this, it suffices to check on the perfectoid basis for the underived version: we have $C_{\text{cts}}(S, \mathrm{R}^+) = \underline{\mathrm{Hom}}(\mathbf{Z}[S], \underline{\mathrm{R}^+})$. Then using that perfectoids form a basis and deriving both sides, in general

$$\mathrm{R}\Gamma((\mathrm{X}\times S)_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}) \simeq \mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}[S], \mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})).$$

So we then apply this to the case of G. Once we have seen that the totalization of $\mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}[\mathrm{G}^\bullet], \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}))$ computes $\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})$ we are done, since the first totalization is how one computes the homotopy fixed points $(-)^{\mathrm{hG}}$.

For applications involving condensed enhancements of comparison theorems involving $\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+)$, it is extremely useful to understand the relation to classical pro-étale cohomology.

Recall that we have a derived p-completion functor

$$\Gamma^*(-)^\wedge_p : \mathrm{D}(\mathsf{Ab}) \to \mathrm{D}(\mathrm{Cond}(\mathsf{Ab}))$$

sending $\mathrm{A} \mapsto \mathrm{R}\varprojlim \underline{\mathrm{A}}/^{\mathbb{L}}p^n$.

Remark 1.14. We call this Γ^* because $\Gamma_* : \mathrm{D}(\mathrm{Cond}(\mathsf{Ab})) \to \mathrm{D}(\mathsf{Ab})$ evaluating on a point is right adjoint to Γ^*. Note that both are exact functors, and so we can use the same notation for their derived versions.

Theorem 1.15. Let X be a rigid affinoid space over a mixed characteristic non-archimedean field. There is an isomorphism

$$\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}) \simeq (\Gamma^*\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \widehat{\mathcal{O}}^+))^\wedge_p.$$

Proof. First, we note that one can actually upgrade the previous construction of an affinoid perfectoid cover. This is because given an affinoid perfectoid space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$, we can take a cofiltered limit of all affinoid étale covers of $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$. Modulo set theoretic issues, this gives us a *strictly totally disconnected* perfectoid space which is pro-étale over $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$. The property of being strictly totally disconnected means that all étale covers split (has a section) and that the space is quasicompact.

Calling the strictly totally disconnected perfectoid space $\tilde{X}$, we have pro-étale covers

$$\tilde{X} \to \mathrm{Spa}(R, R^+) \to X$$

and hence we can localize to the totally disconnected perfectoid situation.

The advantage of this is that now we literally have $H^i(\tilde{X}_{\text{proét}}, \widehat{\mathcal{O}}^+) = 0$ for $i > 0$. This can be shown by observing that global sections in the étale topology are exact when we work with a totally disconnected perfectoid space. Indeed, given a surjection of étale sheaves $f : \mathscr{F} \to \mathscr{G}$ we see that given a global section s of $\mathscr{G}$ we see by definition there's an étale cover $c : \tilde{X}' \to \tilde{X}$ and a global section $t \in \mathscr{F}(\tilde{X}')$ so $f(t) = c^* s$. But then c has a section σ, which means that $f(\sigma^*(t)) = s$ (we have $\sigma^* f(t) = f(\sigma^*(t))$). In particular, f is surjective on global sections, as $\sigma^*(t)$ is a global section of $\tilde{X}$.

We can compare the pro-étale and étale topology modulo p to obtain

$$R\Gamma(\tilde{X}_{\text{proét}}, \widehat{\mathcal{O}}^+/p^n) \simeq R\Gamma(\tilde{X}_{\text{ét}}, \mathcal{O}^+/p^n).$$

The latter is concentrated in degree zero, as global sections are exact in the étale topology on a totally disconnected perfectoid space. In this setting, the exact same reasoning applies for $\widehat{\mathcal{O}}^+_{\text{cond}}$ verbatim: the argument we gave earlier also applies to condensed sheaves on $\tilde{X}_{\text{ét}}$.

Now we are ready to apply descent. Observe that every term in the Cech nerve remains strictly totally disconnected (roughly the reason is that fibres of a pro-étale perfectoid cover over $\tilde{X}$ are profinite sets). Thus, applying descent one obtains explicitly that

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}) \simeq \lim_{[n] \in \Delta} \widehat{\mathcal{O}}^+_{\text{cond}}(\tilde{X}^{(n)}).$$

Now by definition, we obtain

$$\widehat{\mathcal{O}}^+_{\text{cond}}(\tilde{X}^{(n)}) := \underline{\widehat{\mathcal{O}}^+(\tilde{X}^{(n)})} \simeq \Gamma^* \widehat{\mathcal{O}}^+(\tilde{X}^{(n)})^\wedge_p,$$

where the last assertion holds since $\underline{\widehat{\mathcal{O}}^+}$ carries the p-adic topology. The claim then follows. □