

A PRISMATIC RIEMANN-HILBERT CORRESPONDENCE

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ABSTRACT. We show that there is an antiequivalence of categories between constructible p -Kummer-étale $\mathbf{Z}_p$ -sheaves on the log adic generic fiber $(X, \mathcal{M})_\eta$ for the logarithmic stratification induced by a horizontal normal crossings divisor D and certain finitely generated unit Laurent crystals on the log prismatic site. We verify p^+ -perverse t -exactness, give a local description of finitely generated unit crystals on a framed log affine in terms of (φ, Γ) -modules, and show pushforward compatibility for strict log smooth proper morphisms.

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1. INTRODUCTION

Let k be a perfect field of characteristic p . For smooth k -schemes, Emerton and Kisin gave a contravariant Riemann-Hilbert correspondence describing constructible $\mathbf{F}_p$ sheaves using finitely

generated unit Frobenius modules in [EK04b]. Their result is a contravariant generalization of Katz's correspondence in [Kat73], which describes the full subcategory of étale local systems.

In mixed characteristic, Katz's theorem plays an important role in describing local systems on the adic generic fiber of a p -adic formal scheme using (φ, Γ) modules. This idea is originally due to Fontaine in [Fon90], and later generalized by Andreatta-Iovita in [AI08]. Using modern prismatic methods, globalizations were given in [MW25] and [BS23] using Laurent F-crystals.

In this article we give a Riemann-Hilbert correspondence in mixed characteristic extending these results to describe constructible sheaves on a fixed normal crossings stratification. Roughly speaking, our construction uses the Emerton-Kisin correspondence in the role played by Katz's correspondence in the theory of (φ, Γ) -modules.

We work in a logarithmic setting: in our setup, p is odd and $X/\mathrm{Spf} W(k)$ is a smooth quasicompact p -adic formal scheme, with log structure $\mathcal{M} = \mathcal{M}_D$ induced by a horizontal normal crossings divisor D . Letting $\rho(\bar{x}) = \mathrm{rk}_{\mathbb{Z}} \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$ for geometric points $\bar{x}$ of the adic generic fiber X_η , the induced logarithmic strata on the log adic generic fiber $(X, \mathcal{M})_\eta$ are the connected components of the loci where $\rho = a$. We write $D_{\mathrm{cons}, D}^{(b)}((X, \mathcal{M}_D)_\eta, \mathbb{Z}_p)$ for Kummer-étale complexes constructible along these strata. As in the correspondences for Kummer-étale local systems of [IKY26] and [KL16, Theorem 7.5.4], objects $\mathcal{F}$ are p -Kummer when the prime-to- p log inertia $I_{\bar{x}}^{\log, (p)'}$ acts trivially on $\mathcal{F}_{\bar{x}}$ at every log geometric point $\bar{x}$ (Definition 4.23).

We let $(X, \mathcal{M})_\Delta$ denote the absolute log prismatic site of $(X, \mathcal{M})$, and $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$ the sheaf assigning to a log prism $(A, I, M_{\mathrm{Spf} A})^a$ the ring $A[1/I]_p^\wedge$. In Definition 4.3 we introduce the category $D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ of finitely generated unit Laurent crystals, which are certain crystals $\mathcal{E}$ over $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$ equipped with an isomorphism $\varphi^* \mathcal{E} \simeq \mathcal{E}$ and a finiteness condition similar to [EK04b]. The main result of this article is then the following.

Theorem A (Theorem 5.29, Corollary 5.33). *Consider $(X, \mathcal{M}_D)$ with X smooth and quasicompact over $\mathrm{Spf} W(k)$ as above. There is a symmetric monoidal equivalence*

$$T_{\mathrm{ét}} : D_{\mathrm{fgu}}((X, \mathcal{M}_D)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\mathrm{op}} \longrightarrow D_{\mathrm{cons}, D}^{(b), \mathrm{pKum}}((X, \mathcal{M}_D)_\eta, \mathbb{Z}_p)$$

where the target consists of the p -Kummer complexes in $D_{\mathrm{cons}, D}^{(b)}((X, \mathcal{M}_D)_\eta, \mathbb{Z}_p)$. If X_η is pure of dimension d , then $T_{\mathrm{ét}}[d]$ identifies the standard t -structure on the source with the p^+ -perverse t -structure on the target.

One does not need to check on all prisms to verify the finitely generated unit condition. It in fact suffices to check on a covering family of logarithmic q -de Rham prisms, as in Remark 4.15. The theorem also holds over $W(k)[\zeta_{p^\ell}]$ for $\ell \geq 0$, as explained in Corollary 5.33.

We can also study ordinary constructible sheaves within this correspondence. Indeed there is a morphism of sites

$$\epsilon : (X, \mathcal{M})_{\eta, \mathrm{prokét}} \longrightarrow X_{\eta, \mathrm{proét}}$$

whose pullback ϵ^* is fully faithful on constructible sheaves and lands in the p -Kummer sheaves in the heart of $D_{\mathrm{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbb{Z}_p)$. Thus ordinary constructible sheaves (for the stratification induced by D) form a full subcategory at the level of hearts. See Remark 5.32 for the corresponding condition on the crystal side.

Finitely generated unit Laurent crystals also admit an explicit local description in terms of (φ, Γ) -modules. We state the variant over $W(k)$, although we can handle $W(k)[\zeta_{p^\ell}]$ as well. In what follows a *framed log affine* is a $W(k)$ -algebra R that is p -completely étale over $W(k)\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$ with log structure $\mathcal{M}$ induced by $D = V(T_1 \dots T_r)$.

Theorem B (Variant of Theorem 4.7). *Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine, with framing denoted by $\square$. Evaluation on the log q -de Rham prism $(R^\square[[q-1]], [p]_q, \mathbf{N}^r)^a$ over R induces an equivalence*

$$D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p) \simeq D_{\mathrm{fgu}}(R^\square((q-1))_p^\wedge)^{h\Gamma},$$

where $(a, b_1, \dots, b_n) \in \Gamma = \mathbf{Z}_p^\times \ltimes \mathbf{Z}_p(1)^n$ acts by $q \mapsto q^a$ and $T_i \mapsto q^{b_i} T_i$, and “fgu” or finitely generated unit means that the complex has a unit Frobenius and is perfect over $R^\square((q-1))_p^\wedge \langle F \rangle$.

We also use this to make $T_{\mathrm{ét}}$ completely explicit locally in Lemma 4.30 (although in principle the definition in Definition 4.20 is already computable). In this formalism it is relatively straightforward to write down some simple examples. In the case of $\mathrm{Spf} R = \mathrm{Spf} W(k)\langle T \rangle$ with $\Gamma = \mathbf{Z}_p^\times \ltimes \mathbf{Z}_p(1)$ and log structure $\mathcal{M}_{V(T)}$, we describe the objects corresponding to $j_! \mathbf{Z}_p$ and $i_* \mathbf{Z}_p$ under Theorem 4.7. Here j is the inclusion of the punctured log disk $\mathbb{D}^\times \rightarrow (X, \mathcal{M}_{V(T)})_\eta$, and i the complementary inclusion of the origin (with its pullback log structure).

In the notation of Theorem B, $M = R^\square((q-1))^\wedge [1/T]_p^\wedge$ corresponds to $j_! \mathbf{Z}_p$. We equip it with an action by $\Gamma = \mathbf{Z}_p^\times \ltimes \mathbf{Z}_p(1)$ where $q \mapsto q^x$ for $x \in \mathbf{Z}_p^\times$ and $T \mapsto q^{p^x} T$ for $x \in \mathbf{Z}_p(1)$. Moreover it has a unit Frobenius, sending $T \mapsto T^p$ and $q \mapsto q^p$. The (φ, Γ) -module $R^\square((q-1))^\wedge [1/T]_p^\wedge / R^\square((q-1))^\wedge$ corresponds to $i_* \mathbf{Z}_p[-1]$ (for the unshifted version of $T_{\mathrm{ét}}$). In Example 5.35, we give a more detailed calculation corresponding to a constructible sheaf (which has a nontrivial gluing datum between strata).

Our third main result is pushforward compatibility.

Theorem C (Theorem 6.11). *Let $f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ be a strict log smooth proper morphism of pure relative dimension d between smooth quasicompact p -adic formal schemes over $W(k)[\zeta_{p^\ell}]$ for some $\ell \geq 0$, and let D_X and D_Y be horizontal normal crossings divisors inducing $\mathcal{M}$ and $\mathcal{N}$. Suppose the adic generic fibers have pure dimensions d_X and d_Y . Then there is a commutative diagram*

$$\begin{array}{ccc} D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p)^{\mathrm{op}} & \xrightarrow{T_{\mathrm{ét}, X}[d_X]} & D_{\mathrm{cons}, D_X}^{(b)}((X, \mathcal{M})_\eta, \mathbf{Z}_p) \\ \downarrow Rf_* \{d\}[d] & & \downarrow Rf_{\eta,*} \\ D_{\mathrm{fgu}}((Y, \mathcal{N})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p)^{\mathrm{op}} & \xrightarrow{T_{\mathrm{ét}, Y}[d_Y]} & D_{\mathrm{cons}, D_Y}^{(b)}((Y, \mathcal{N})_\eta, \mathbf{Z}_p). \end{array}$$

The shift and twist arise from a logarithmic Poincaré duality result for perfect Laurent crystals proved in Corollary 6.7. For the Laurent F-crystal equivalences in either [BS23] or [IKY26], the analogous pushforward compatibility theorem does not appear in the literature. The problem is that it is not obvious that the right adjoint Rf_* on Laurent crystals preserves $\mathrm{Perf}(X_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p)^{\varphi=1}$. For this reason existing pushforward compatibility results (such as the result stated in [GL25]) assume that the Laurent F-crystal arises from a prismatic F-crystal, as in that situation perfectness can be deduced from the Hodge-Tate comparison. As Theorem 6.11 is compatible with a contravariant version of [IKY26], we also obtain pushforward compatibility for this correspondence (as well as the one in [BS23, Corollary 3.7] as a degenerate case).

There are also other compatibilities we prove through more straightforward arguments. We show pullback compatibility in Corollary 4.18, and Theorem 5.4 gives compatibility with $j_!$ where j is the inclusion of $X_\eta \setminus D_\eta$. Locally, through Kashiwara's lemma (Lemma 5.19) we also give compatibility with i_* and $i^!$. Finally, due to the cohomological smoothness results in [Kip26a] we also have compatibility with $f^!$ for strict log smooth proper maps of pure relative dimension d by combining pushforward compatibility with $f^! \mathcal{O}_\Delta[1/I_\Delta]^\wedge \simeq \mathcal{O}_\Delta[1/I_\Delta]^\wedge \{d\}[2d]$ from Lemma 6.4.

Related work. While not directly providing Riemann-Hilbert correspondences, some of the motivation for this work came from comparison theorems in [Fal89] and [Yam11]. The article [Fal89] in fact originally motivated Emerton and Kisin to produce [EK04b] in the hope of obtaining a result of the type in Theorem A, and our correspondence shares a number of similarities with what Faltings does in this article. The comparison theorems in [Fal89] allow various support conditions related to a normal crossings divisor D , and point towards it being fruitful to work in a logarithmic setting with the stratification induced by D just as we do in this article (indeed Faltings uses a somewhat similar logarithmic setup). Various other similarities arise, for example an analogue of the p -Kummer condition in [Fal89, II(i)] and some constructions analogous to Remark 5.32 in [Fal89, III(d)]. It seems plausible there could be a more direct relationship, which we speculate further about in Remark 5.6.

There are also several related works which are more recent. Bhatt and Lurie construct a covariant p -adic Riemann-Hilbert functor, described in [Bha25, §1.1(4) and Theorem 6.3.2]. Let $K/\mathbb{Q}_p$ be a nonarchimedean field of finite p -cohomological dimension. For a p -adic formal scheme X locally of finite type over $\mathcal{O}_K$, it takes the form

$$\mathrm{RH}_\Delta : D_{p\text{-comp}}((X_\eta)_{\text{ét}}, \mathbf{Z}_p)^{\mathrm{oc}} \longrightarrow D(X^{\Delta, \mathrm{pfd}})^{a, \varphi=1}.$$

Here the source consists of derived p -complete overconvergent sheaves, $X^{\Delta, \mathrm{pfd}}$ is the perfect prismaticization, and ' a ' denotes passage to almost coefficients. Their functor is fully faithful, symmetric monoidal, and identifies the dualizable subcategories. It is moreover compatible with the positive characteristic correspondence of [BL19] as explained in [Bha25, Theorem 6.2.3]. They also give an algebraic variant without almost coefficients in [Bha25, Variant 1.1.9]. When K is perfectoid, for Zariski-constructible inputs Bhatt and Lurie prove almost coherence and proper pushforward compatibility results, as well as perverse t -exactness with finite coefficients [Bha25, Theorem 7.3.6, Corollary 7.3.7, and Theorem 10.3.4].

Mann constructs in [Man22, Theorem 3.9.23] a related fully faithful functor from overconvergent étale $\mathbb{F}_p$ -complexes on small v -stacks to solid almost $\mathcal{O}^+/\pi$ -modules with unit Frobenius, where $\pi \mid p$ is a pseudouniformizer. It identifies the perfect subcategories and forms part of a six-functor formalism yielding proper pushforward comparisons (under suitable hypotheses) and Poincaré duality in [Man22, Corollary 3.9.24 and Theorem 3.10.20].

We point out that our correspondence has two practical advantages when compared to these very general Riemann-Hilbert functors. The first is that we can locally formulate the crystal side of the correspondence as a generalization of (φ, Γ) -modules, which is then amenable to computation as we work over a deperfection base (see Example 5.35). The second is that the functor is an *equivalence* onto the full subcategory of p -Kummer constructible sheaves, so there is a clear and easily verifiable condition on the crystal side corresponding to constructibility. While we do not pursue it here, we expect our correspondence (up to a duality) to be realized as a deperfection of

a logarithmic variant of the Bhatt-Lurie correspondence describing overconvergent p -Kummer sheaves.

The fact that the correspondence is deperfection makes it in principle possible to access C_{st} -type comparison theorems for constructible sheaves, which we hope to pursue in future work. This sort of application does not seem possible with other Riemann-Hilbert correspondences due to taking perfections. More is said about this in Remark 5.6.

Overview of the proofs. All arguments work first over $W(k)[\zeta_p]$, and then we extend to other bases once Theorem A is proved.

In Definition 4.3 we introduce the category of finitely generated unit Laurent crystals (which we denote by $D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$), and later construct a functor

$$T_{\text{ét}} : D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\text{op}} \longrightarrow D((X, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p).$$

Theorem A requires us to show this is fully faithful and determine the essential image as the p -Kummer constructible sheaves along D .

We work in the logarithmic setting, so the natural analogue of étale descent is strict étale descent. The main idea behind proving Theorem A is to prove strict étale descent of both sides and then locally use the explicit (φ, Γ) -module description in Theorem B.

We defer explaining both of these results, and instead focus on how one proves the main theorem granting the local (φ, Γ) -module description in Theorem B. In Lemma 4.30, we show that $T_{\text{ét}}$ has a local description similar to how one constructs the functor for ordinary (φ, Γ) -modules – from this one can see that the functor lands in the p -Kummer subcategory. The task is then to explain where constructibility along D comes from.

Using [Bli08], we deduce that modulo p any finitely generated unit (φ, Γ) -module has a presentation $M \simeq \text{colim}_n \varphi^{*n} M_0$ where M_0 is a coherent Γ -module with a Γ -equivariant injection or “root map” $\beta : M_0 \rightarrow \varphi^* M_0$. In Lemma 4.26 we classify possible Fitting ideals for M_0 , $\varphi^* M_0$ and $\text{coker } \beta$, allowing one to conclude that on each stratum induced by D the root map is an isomorphism. Recalling Katz’s equivalence one deduces the result is lisse on every log stratum.

The argument for full faithfulness and essential surjectivity also proceeds by examining these strata, the most important being the open stratum $U_{\eta} := X_{\eta} \setminus D_{\eta}$. This part we also do globally in Theorem 5.4. We begin by constructing a crystal $\mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}(*D)$ which is sent to $j_! \mathbf{Z}_p$ by $T_{\text{ét}}$. Deducing essential surjectivity uses the results of [IKY26], as one can use Lütkebohmert’s theorem as recast in [DLLZ23, Corollary 6.3.4] to extend a local system on U_{η} to a Kummer-étale local system. After a finite prime-to- p Kummer cover $(X_N, \mathcal{M}_N)$ with Galois group μ_N^r on the open stratum, this extended local system becomes p -Kummer and has a corresponding Laurent F -crystal $\mathcal{E}$. We tensor with $\mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}(*D)$, and keeping track of μ_N^r descent data, descend back to a Laurent crystal on $(X, \mathcal{M})_{\Delta}$. Full faithfulness on the open stratum uses a decompletion statement that base change to the perfection is fully faithful; Remark 5.13 explains why this does not follow from Frobenius alone. We must also understand full faithfulness for analytification on perfectoid spaces, where we use [Bha25].

To work towards the local version of Theorem A one proves a variant of Kashiwara’s lemma in Lemma 5.19. This allows us to handle closed strata with their hollow pullback log structures, and we study a variant of the open stratum result with hollow log structures locally. Full faithfulness

follows by identifying thick generators corresponding to local systems supported on locally closed strata and checking full faithfulness for maps between these: the remaining case is roughly to understand maps between local systems supported on closed and open strata. This is handled in Lemma 5.24 and Lemma 5.25 on each side of the correspondence, which allow us to reduce this assertion to an RHom on a single stratum. The general outline of the argument is summarized in the toy model in the beginning of §5.2.

Now we return to the two ingredients needed to reduce the global theorem to a theorem about (φ, Γ) -modules. The first claim is Theorem B expressing $D_{\mathrm{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ locally via (φ, Γ) -modules. This is handled in §3. Proposition 3.13 shows that evaluation on the log q -de Rham prism yields a fully faithful functor into semilinear Γ -modules. For essential surjectivity in Theorem 4.7 one uses roots to present finitely generated unit (φ, Γ) -modules. Modulo p , [Bli08] gives a unique minimal coherent root in the sense of Definition 4.5. Minimality forces this to be Γ -equivariant, and we use Theorem 3.5 to carry a Γ -equivariant root to a coherent Laurent crystal. This shows essential surjectivity modulo p , which is sufficient for the argument.

The second ingredient is strict étale descent of $D_{\mathrm{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. We work with a solid variant $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ of Laurent crystals, explained in Definition A.11. The appendix proves that the logarithmic Breuil-Kisin atlases are completely descendable in the sense of [Kip26b, Definition 2.4.1.7]. Consequently, ordinary solid Laurent crystals are computed by the Čech nerve of the log q -de Rham prism. We also show strict étale descent using similar methods. One must then show that the finitely generated unit condition itself is strict étale local. We do this by again using roots: the main point is to promote these minimal roots *as modules* from log q -de Rham evaluations to minimal roots in *coherent Laurent crystals* on framed log affines. Once this is done, their uniqueness as subcrystals makes them glue and proves descent (since coherent Laurent crystals form a stack, using [Mat22]).

Finally, for pushforward compatibility in Theorem 6.11, we first develop [Kip26b] and [Kip26a] to construct a stacky approach to solid Laurent crystals in the log setting and equip them with a six functor formalism. The idea of the argument is to show that the pushforward $Rf_*\{d\}[d]$ preserves the finitely generated unit condition, which formally yields the desired compatibility. To deduce this preservation one again uses roots: it can be checked modulo p , and $Rf_*\mathcal{E} \simeq Rf_!(\mathrm{colim}_n \varphi^{*n}\mathcal{E}_0) \simeq \mathrm{colim}_n Rf_*\varphi^{*n}\mathcal{E}_0$. Finally we show that $Rf_*\varphi^{*n}\mathcal{E}_0 \simeq \varphi^{*n}Rf_*\mathcal{E}_0$ using a variant of Cartier descent, yielding $Rf_*\mathcal{E} \simeq \mathrm{colim}_n \varphi^{*n}Rf_*\mathcal{E}_0$. This last presentation shows the result is finitely generated unit.

Notation. All categories are ∞ -categories, unless otherwise stated. We will generally assume X is a smooth quasicompact p -adic formal scheme over $\mathrm{Spf} W(k)[\zeta_{p^\ell}]$, $\ell \geq 0$, with log structure $\mathcal{M} = \mathcal{M}_D$ induced by a horizontal normal crossings divisor D .

Our convention is that $(X, \mathcal{M})_{\Delta}$ denotes the strict absolute log prismatic site (there are other variants, e.g. the saturated site, we will not use, but see A.10 for why this does not matter). We use the term ‘Laurentization’ for the passage $(A, I) \mapsto A[1/I]_p^{\wedge}$ for a prism (A, I) or log prism. We write $D_{\blacksquare}$ and D_{fgu} for solid Laurent crystals and finitely generated unit Laurent crystals, respectively (Definitions A.11 and 4.3). We use $T_{\mathrm{ét}}$ for the étale realization in Theorem A (see Definition 4.20).

We also use several stacks: $(X, \mathcal{M})^\Delta$ denotes the log prismatic stack (see A.1), and we also use $(X, \mathcal{M})^{\Delta, \blacksquare}$ and $(X, \mathcal{M})^{\mathcal{L}, \blacksquare}$ for log prismatic and Laurent analytic stacks, following the constructions in [Kip26b]. The Laurent stack is constructed to capture $D_{\blacksquare}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ as $D_{\blacksquare}((X, \mathcal{M})^{\mathcal{L}, \blacksquare})$.

We will frequently reduce to the situation of a framed log affine, as in Definition 3.2. This has an associated log q -de Rham prism we will use, with associated Laurentization $A = R_0^\square((q-1))_p^\wedge$. We generally denote the ring of the actual prism by A^+ . We use $\square$ to denote the framing, as it determines the Frobenius and action of $\Gamma = (1 + p\mathbb{Z}_p) \ltimes \mathbb{Z}_p(1)^n$ (or other variants depending on the base). We often write $\bar{A} = A/p$, and use $(-)^{h\Gamma}$ for the totalization of the continuous cobar construction.

We distinguish mapping complexes RHom from internal derived Hom RHom . Derived reduction is $M//f := \mathrm{cofib}(f : M \rightarrow M)$, with several elements denoting iterated cofibers; single slashes denote underived quotients.

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Statement on the use of AI. AI tools were used to check for mathematical errors during and after writing this article.

AI tools also assisted in some of the arguments in this paper: Lemma 3.12 and its helper lemma 3.10, the helper lemmas for Proposition 5.9, and in simplifying the author's first counterexample in Remark 5.13. For example, in the helper lemmas for Proposition 5.9, the idea of using techniques from [KL16, §5.6] for decompletion is due to the author. AI tools worked out the explicit contracting homotopies needed to actually carry out this approach, and the author carefully checked this and simplified the argument.

Some of these influence later arguments, for example ideas from Lemma 3.12 and Lemma 3.10 are used again in Lemma 6.8 to make the final calculation.

AI tools were also used for literature search, which resulted in several useful suggestions. Many of these are simply locating references for standard statements, but some had more impact. The ones which had the most impact were locating [Wig73], which removed a delicate point-set topology argument, and the papers [HM24] and [Mik25] when asked about six functor formalism results in the literature to check properness.

2. THE EMERTON-KISIN CORRESPONDENCE

In this section, we will review (and slightly generalize) some results about the Emerton-Kisin correspondence in [EK04b].

As a special case of the Emerton-Kisin correspondence for lisse sheaves, we begin with Katz's correspondence which was introduced in [Kat73, Proposition 4.1.1]. There the result was proved for a regular $\mathbb{F}_p$ -algebra, but actually this was not an essential ingredient in the proof as observed in [BS23, Proposition 3.4]. Indeed, the result holds for a general $\mathbb{F}_p$ algebra.

However the version in [BS23, Proposition 3.4] only describes $D_{\mathrm{lisse}}^{(b)}(\mathrm{Spec} R, \mathbb{F}_p)$, and we will need a variant for $\mathbb{Z}_p$ coefficients in the case R is a perfect $\mathbb{F}_p$ -algebra. We write $D_{\mathrm{lisse}}^{(b)}(\mathrm{Spec} R, \mathbb{Z}_p)$ for

full subcategory of $D(\mathrm{Spec} R_{\mathrm{pro\acute{e}t}}, \mathbf{Z}_p)$ consisting of locally bounded derived p -complete complexes of sheaves $\mathcal{F}$ such that $\mathcal{F} // p$ has all lisse cohomology sheaves.

Theorem 2.1 (Katz). *Let R be a perfect $\mathbf{F}_p$ -algebra. Then there is an equivalence of categories*

$$\mathrm{Perf}(W(R))^{\varphi=1} \simeq D_{\mathrm{lisse}}^{(b)}(\mathrm{Spec} R, \mathbf{Z}_p),$$

induced by the solution functor

$$M \longmapsto (M \otimes_{W(R)} W(\mathcal{O}_{\mathrm{Spec} R, \mathrm{pro\acute{e}t}}))^{\varphi=1}.$$

Proof. We apply Lemma 2.3, which is a variant of derived Nakayama for categories that we will use frequently (and prove shortly).

The hypotheses of this lemma ask that both source and target can be written as $\mathbf{N}$ -indexed limits of idempotent complete $\mathbf{Z}_p$ -linear categories. For the source we use $\mathrm{Perf}(W_n(R))^{\varphi=1}$ and for the target we use $D_{\mathrm{lisse}}^{(b)}(\mathrm{Spec} R, \mathbf{Z}/p^n)$ (we impose a Tor boundedness condition now). It is easy to check idempotent completeness here since both sit in ambient idempotent complete categories and their defining conditions are closed under retracts.

Finally, the reduction modulo p becomes [BS23, Proposition 3.4]. Thus by Lemma 2.3 the desired claim follows. $\square$

Remark 2.2. The Emerton-Kisin correspondence is contravariant. Since Perf is self-dual, the previous correspondence also admits a dual or contravariant version. The solution functor is $M \mapsto \underline{\mathrm{RHom}}_{\mathrm{Perf}^\varphi(W(R)_{\mathrm{pro\acute{e}t}})}(M_{\mathrm{pro\acute{e}t}}, W(R)_{\mathrm{pro\acute{e}t}})$ (regarded as a pro\acute{e}tale sheaf). Similarly other correspondences reducing to Katz's theorem, such as [BS23, Corollary 3.7] or [IKY26] admit contravariant versions.

We will now prove Lemma 2.3. Before we go ahead with this, we will need some basic setup for the lemma to work. We thank Kush Singhal for suggesting trying Verdier localization, which allowed the author to simplify their first (much messier) proof.

Setup. Let $\mathcal{C}$ be an idempotent complete $\mathrm{Perf}(\mathbf{Z}_p)$ -linear stable ∞ -category. Write

$$\mathcal{C}_n := \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{Z}_p)} \mathrm{Perf}(\mathbf{Z}/p^n).$$

We say that $\mathcal{C}$ is derived p -complete if the natural functor $\mathcal{C} \rightarrow \varprojlim_n \mathcal{C}_n$ is an equivalence.

We ask that functors between derived p -complete categories $F : \mathcal{C} \rightarrow \mathcal{D}$ in this setup are induced by a system of compatible functors $F_n : \mathcal{C}_n \rightarrow \mathcal{D}_n$. Here F is $\mathbf{Z}_p$ -linear and the F_n induced by scalar extension.

Lemma 2.3. *Let $\mathcal{C} = \lim \mathcal{C}_n$ and $\mathcal{D} = \lim \mathcal{D}_n$ be derived p -complete idempotent complete categories as in the previous setup. Consider an exact functor*

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

induced by a compatible system $\{F_n\}$ of functors $\mathcal{C}_n \rightarrow \mathcal{D}_n$.

If F_1 is an equivalence, then F is an equivalence.

Proof. Full faithfulness reduces to the ordinary derived Nakayama lemma. Indeed,

$$\mathrm{RHom}_{\mathcal{C}_1}(X_1, Y_1) \simeq \mathrm{RHom}_{\mathcal{C}}(X, Y//p) \simeq \mathrm{RHom}_{\mathcal{C}}(X, Y) \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{F}_p.$$

Note that by our setup $\mathrm{RHom}_{\mathcal{C}}(X, Y)$ is derived p -complete. Similarly one also gets full faithfulness of F_n .

For essential surjectivity, we observe that F is exact and fully faithful. The essential image will be a thick subcategory of the target $\mathcal{D}$ using idempotent completeness of $\mathcal{C}$; in fact the same is true for each F_n (this point is why we impose idempotent completeness).

Thus we can consider the Verdier quotient

$$\mathcal{Q} = \mathrm{cofib}(F)$$

where cofibers are taken in $\mathrm{Cat}_{\mathbf{Z}_p}^{\mathrm{perf}}$. Set also $\mathcal{Q}_n = \mathrm{cofib}(F_n)$, although a priori it need not be the case $\mathcal{Q} = \lim_n \mathcal{Q}_n$ (and we will not need this).

It will suffice to show each $\mathcal{Q}_n = \mathcal{Q} \otimes_{\mathrm{Perf}(\mathbf{Z}_p)} \mathrm{Perf}(\mathbf{Z}/p^n)$ is zero, using the fact that $\mathcal{Q}_1$ is zero. Because we showed that the essential image in $\mathcal{D}_n$ is thick, this is enough to show essential surjectivity for each n and hence for F . Otherwise, we would only deduce that $F_n(\mathcal{C}_n)$ generates $\mathcal{D}_n$.

Considering some $M \in \mathcal{Q}$, observe that

$$\mathrm{cofib}(p \cdot \mathrm{id}_M) = M \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{Z}/p\mathbf{Z}.$$

But there is an adjunction $\mathrm{red}_1 \rightleftarrows \mathrm{for}_1$ for the reduction functor $\mathrm{red}_1 : \mathcal{Q} \rightarrow \mathcal{Q}_1$ (the right adjoint being forgetful), which has the property that $\mathrm{for}_1 \mathrm{red}_1$ recovers $- \otimes_{\mathbf{Z}_p} \mathbf{Z}/p\mathbf{Z}$. Since in this case red_1 is zero we conclude that $\mathrm{cofib}(p \cdot \mathrm{id}_M)$ is zero.

Thus p acts invertibly on $\mathcal{Q}$ and therefore each $\mathcal{Q}_n$, which can only be the case if each $\mathcal{Q}_n$ is the zero category. The claim then follows. $\square$

Katz's theorem is crucial in proving the equivalence

$$\mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1} \simeq D_{\mathrm{lis}}^{(b)}(X_{\eta}^{\diamond}, \mathbf{Z}_p)$$

for bounded p -adic formal schemes X in [BS23]. Precisely, this means that when $X = \mathrm{Spf} R$ is perfectoid, the result reduces to Katz's equivalence; their theorem is deduced by a reduction to this case. Our result for finitely generated unit Laurent crystals will be compatible in a similar way with the Emerton-Kisin correspondence for constructible sheaves.

We now extend the Emerton-Kisin correspondence of [EK04b] to $\mathbf{Z}_p$ coefficients, deduced from the mod p theorem by Lemma 2.3. We fix a perfect field k of characteristic p . If $\mathcal{X}/\mathrm{Spf} W(k)$ is a smooth formal scheme, we let $D_{\mathcal{X}}$ denote the sheaf of infinitesimal differential operators. Locally, if $X \rightarrow \mathbf{A}_k^n$ is étale with formal lift $\mathcal{X}$, this can be pulled back from the sheaf associated to $D_{\mathrm{Spf} W(k)\langle T_1, \dots, T_n \rangle} = W(k)\langle T_1, \dots, T_n, \partial_1^{[i_1]}, \dots, \partial_n^{[i_n]} \rangle$, where all ∂_i are given full divided powers. We put

$$D_{F, \mathcal{X}} := \widehat{\bigoplus_{i \geq 0} F^{*i} D_{\mathcal{X}}}.$$

Modules over this noncommutative sheaf are infinitesimal crystals $\mathcal{E}$ with semilinear Frobenius $\varphi_{\mathcal{E}}$. There is a useful Taylor formula comparing Frobenius lifts makes $D_{F, \mathcal{X}}$ canonical and global; see [EK04a, §6.1-6.2].

There is also a perverse t -structure that the Emerton-Kisin correspondence is compatible with. We define this as follows, extending [Gab04]. Strictly speaking this only produces a t -structure on $D_{\text{cons}}^{(b)}(X, \mathbf{F}_p)$. However we can define a middle perversity function as $p(x) = -\dim \overline{\{x\}}$. We set the ≤ 0 part of the perverse t -structure to be constructible $\mathcal{F}$ such that

$$H^i(i_x^{-1}\mathcal{F}) = 0, \quad i > p(x)$$

and the ≥ 0 part as

$$H^i(i_x^!\mathcal{F}) = 0, \quad i < p(x).$$

The heart of this t -structure is $\text{Perv}(X_{\text{proét}}, \mathbf{Z}_p)$.

Definition 2.4. We use $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ to denote Gabber's perverse t -structure on $D_{\text{cons}}^{(b)}(X, \mathbf{Z}_p)$, and write ${}^pH^i$ for perverse cohomology sheaves.

We may then construct the p^+ -perverse t -structure, specified by

$$\begin{aligned} {}^{p^+}D^{\leq 0} &= \{\mathcal{F} \in {}^pD^{\leq 1} \mid {}^pH^1(\mathcal{F}) \text{ is locally bounded } p\text{-power torsion}\}, \\ {}^{p^+}D^{\geq 0} &= \{\mathcal{F} \mid \mathcal{F} // p \in {}^pD^{\geq 0}(X_{\text{proét}}, \mathbf{F}_p)\}. \end{aligned}$$

For every $n \geq 1$, derived reduction modulo p^n is left t -exact for p^+ -perversity. Moreover, $\mathcal{F} \in {}^{p^+}D^{\geq 0}$ if and only if $\mathcal{F} // p \in {}^pD^{\geq 0}(X_{\text{proét}}, \mathbf{F}_p)$. This is similar in spirit to [BH22, Construction 4.3 and Proposition 4.6].

With this, we can now extend the Emerton-Kisin correspondence to $\mathbf{Z}_p$ coefficients. We use the same convention as [HRS23] for constructible sheaves, so that in our situation for $\mathbf{Z}/p^n$ they become the same as bounded constructible Tor finite sheaves. We will not concern ourselves with boundedness versus local boundedness in what follows, as X is assumed quasicompact.

In this setting one defines $D_{\text{fgu}}^b(D_F, \mathcal{X})$ as the full subcategory of $D_{\text{unit}}(D_F, \mathcal{X})$ (i.e. $F^*M \simeq M$) which are bounded and have cohomology sheaves finitely generated over $D_F, \mathcal{X}$.

Theorem 2.5. *Let $\mathcal{X} / \text{Spf } W(k)$ be smooth and quasicompact. Then there is an equivalence*

$$\text{RSol} : D_{\text{fgu}}^b(D_F, \mathcal{X})^{\text{op}} \simeq D_{\text{cons}}^b(X, \mathbf{Z}_p),$$

where $X = \mathcal{X}_{p=0}$. Explicitly, the functor is given by

$$\text{RSol}_X(M) = \underline{\text{RHom}}_{D_F, \mathcal{X}_{\text{proét}}} (M_{\text{proét}}, \mathcal{O}_{\mathcal{X}, \text{proét}}) \in D(X_{\text{proét}}, \mathbf{Z}_p).$$

If X is pure of dimension d , then $\text{RSol}_X[d]$ identifies the standard t -structure on the source with the p^+ -perverse t -structure on the target.

Proof. We first note that one has

$$D_{\text{fgu}}^b(D_F, \mathcal{X}) \simeq \lim_n D_{\text{fgu}, \text{tf}}^b(D_F, \mathcal{X}_{p^n=0})$$

and similarly

$$D_{\text{cons}}^b(X, \mathbf{Z}_p) \simeq \lim_n D_{\text{cons}}^b(X_{\text{proét}}, \mathbf{Z}/p^n) = \lim_n D_{\text{ctf}}^b(X_{\text{proét}}, \mathbf{Z}/p^n).$$

Here tf means Tor finite. The second equivalence follows from [HRS23, Theorem 1.6(2)], and their Remark 7.2 to identify with $D_{\text{ctf}}^b(X_{\text{proét}}, \mathbf{Z}/p^n)$.

Because each side here sits in an obvious ambient category that is idempotent complete and the imposed conditions are stable under retracts, we get the idempotent completeness requirement for Lemma 2.3. It is straightforward to show that the categories we have written here in each limit are obtained by applying $\otimes_{\text{Perf}(\mathbf{Z}_p)} \text{Perf}(\mathbf{Z}/p^n)$.

Moreover, [EK04b, Proposition 15.4.1 and Equation (16.1.3)] give

$$\text{RSol}_X(M) // p^n \simeq \text{RSol}_{X, \mathbf{Z}/p^n}(M // p^n).$$

Thus the solution functor is naturally obtained by a compatible system of solution functors modulo p^n . Since for the mod p reduction we have

$$D_{\text{cons}}^b(X_{\text{proét}}, \mathbf{F}_p) \simeq D_{\text{cons}}^b(X_{\text{ét}}, \mathbf{F}_p)$$

by [HRS23, Theorem 1.6(1)], [EK04b, Corollary 16.2.8] proves the equivalence by applying Lemma 2.3.

The perverse t -exactness claim remains. However, we gave reduction criteria in Definition 2.4, so our strategy will be to reduce to checking it modulo p where all the work is done in [EK04b].

Perverse t -exactness in [EK04b] is actually proved for a variant with $\mathcal{O}_{F,X}$ -modules. This is not an issue, however. One notes that modulo p [EK04b, Proposition 15.4.3] identifies the bounded derived category of finitely generated unit $D_{F,X}$ -modules with the usual finitely generated unit $\mathcal{O}_{F,X}$ -modules, and [EK04b, 16.1.5] identifies their solution functors. In particular we may work with $\mathcal{O}_{F,X}$ -modules, even for perverse t -exactness, because the identification leaves the underlying $\mathcal{O}_{F,X}$ -module unchanged (more precisely the identification is t -exact).

Assume now that X has pure dimension d , and consider $\text{RSol}_X(M)[d]$. By derived Nakayama, $M \in D^{\leq 0}$ if and only if $M // p \in D^{\leq 0}$. This compatibility with $-//p$ and perverse t -exactness modulo p from [EK04b, Theorem 11.5.4] (which applies to $\mathcal{O}_{F,X}$ -modules) identifies the condition $M \in D^{\leq 0}$ with $(\text{RSol}_X(M)[d]) // p \in {}^p D^{\geq 0}$, which holds if and only if $\text{RSol}_X(M)[d] \in {}^{p^+} D^{\geq 0}$ by Definition 2.4. Thus RSol identifies these halves of the two t -structures. Orthogonality identifies the other halves, and thus proves the claim. $\square$

There is also a variant for perfect rings. In this setting we use the definition from [BL19] to make sense of the finitely generated unit condition: one asks for modules to be in $\text{Thick}(W(R)\langle F \rangle_p)$ and to be unit.

Theorem 2.6. *Let R be a perfect $\mathbf{F}_p$ -algebra. Then there is an equivalence*

$$\text{RSol} : D_{\text{fgu}}(W(R))^{\text{op}} \simeq D_{\text{cons}}^b(\text{Spec } R, \mathbf{Z}_p)$$

given by $\text{RHom}_{D^\varphi(W(R))_{\text{proét}}} (M_{\text{proét}}, W(R)_{\text{proét}}) \in D((\text{Spec } R)_{\text{proét}}, \mathbf{Z}_p)$. The equivalence is moreover symmetric monoidal.

Proof. This is essentially already proven in [BL19], but they work largely with $\mathbf{F}_p$ coefficients. The argument is similar to the previous case to make the extension to $\mathbf{Z}_p$ coefficients. Both sides are stable subcategories of their ambient derived categories closed under retracts, hence are idempotent complete and $\text{Perf}(\mathbf{Z}_p)$ -linear; the dual action of $\text{Perf}(\mathbf{Z}_p)$ on the source makes RSol linear. Their reductions (in the sense of the setting of Lemma 2.3) are the perfect unit categories over $W_n(R)$ and the constructible complexes whose stalks have finite Tor-dimension over $\mathbf{Z}/p^n$.

Applying Lemma 2.3 now reduces the claim to [BL19, Theorem 11.4.4], together with [BL19, Theorems 12.5.4 and 12.6.1].

Thus all that remains is to check the equivalence is symmetric monoidal. One may essentially copy the argument in [BL19, Theorem 8.4.1]. The formula for RSol gives a lax symmetric monoidal structure:

$$\mathrm{RSol}(M) \hat{\otimes}_{\mathbf{Z}_p}^L \mathrm{RSol}(N) \longrightarrow \mathrm{RSol}(M \hat{\otimes}_{W(R)}^L N).$$

Both sides are derived p -complete, so derived Nakayama reduces the assertion that this map is an equivalence to the case modulo p , where it may be checked on geometric stalks (one must use pullback compatibility of the correspondence).

Over an algebraically closed perfect field κ , use the dual or contravariant form of Katz's theorem described in the remark following Theorem 2.1. The fact that this is symmetric monoidal can be checked in the same way as [BL19, Theorem 8.4.1]: the category is equivalent to $\mathrm{Perf}(\mathbf{F}_p)$, and reducing to the heart (which is fine since it thickly generates) one can decompose objects as finite direct sums of the unit where the claim is obvious.

This argument hides that we needed to check another point, namely that we needed to check closure under tensor products. Remark 4.4 reduces us to characteristic p , where [BL19, Theorems 12.4.1 and 12.5.4, Proposition 12.3.8] express every finitely generated unit complex as $D(P)$ for a Frobenius complex P perfect over R , where D denotes a *weak dual*. By [BL19, Construction 11.2.7 and the proof of Proposition 12.5.1], $D(P) \simeq \mathrm{colim}_n \varphi^{n*}(P^\vee)$, so cofinality of the diagonal $\mathbf{N} \rightarrow \mathbf{N}^2$ allows us to deduce $D(P) \otimes_R^L D(Q) \simeq D(P \otimes_R^L Q)$ by indexing the colimit over the diagonal. Since $P \otimes_R^L Q$ is again perfect over R , its weak dual is finitely generated unit by [BL19, Proposition 12.3.11]. $\square$

3. SOME RESULTS ON LAURENT CRYSTALS AND Γ -MODULES

Let $(X, \mathcal{M})$ be a bounded fs log p -adic formal scheme. Associated to $(X, \mathcal{M})$ is the absolute log prismatic site $(X, \mathcal{M})_\Delta$, the opposite category of log prisms over $(X, \mathcal{M})$ (see [Kos22] and [KY25]).

On this site, there is a sheaf of rings $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge : (A, I, M_{\mathrm{Spf} A})^a \mapsto A[1/I]_p^\wedge$. We call prismatic crystals over $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$ *Laurent crystals*. The goal of this section is to study these Laurent crystals in the setting where $(X, \mathcal{M})$ is a *framed log affine*, which will be the standard local situation we reduce to in the article.

Assumption 3.1. Let $X/\mathcal{O}_K$ be smooth and quasicompact, let D be a horizontal normal crossings divisor¹, and let $\mathcal{M} = \mathcal{M}_D$ be the log structure induced by D .

We fix throughout a compatible system $(\zeta_{p^m})_{m \geq 1}$ extending ζ_p , and can use it to identify $\mathbf{Z}_p(1)$ with $\mathbf{Z}_p$ if we choose. In this section we assume p is odd and work over $W(k)[\zeta_p]$. At the end of this section in Corollary 3.21, we explain how to make modifications to obtain the result over $W(k)$ (and also with more roots of unity). The assumption that p is odd can surely be removed, but we do not address it here as the paper is long enough.

The first well-known result we show is that locally $(X, \mathcal{M})$ satisfying Assumption 3.1 look like framed log affines as defined below.

¹By this we mean that $D \cup X_s$ is a normal crossings divisor and no irreducible component of D is contained in the special fiber; equivalently, strict étale locally the pair has the form of Definition 3.2

Definition 3.2. Let K be a complete nonarchimedean field of mixed characteristic $(0, p)$. A framing of $(\mathrm{Spf} R, \mathcal{M})$ over $\mathcal{O}_K$ is a p -completely étale map $\mathcal{O}_K \langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle \rightarrow R$ under which $\mathcal{M}$ is the pullback of the log structure with chart $\mathbf{N}^r$ whose generators map to $T_1, \dots, T_r$. We denote the resulting log formal scheme $\mathrm{Spf} \mathcal{O}_K \langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$ by $\mathbb{T}_r^n$, the framing map by $\square$, and call $(\mathrm{Spf} R, \mathcal{M})$ a *framed log affine*.

Remark 3.3. For a framed log affine, we use n and r as the corresponding integers in Definition 3.2. We sometimes call the étale lifts of $T_1, \dots, T_r$ “boundary coordinates”.

If $K/W(k)[1/p]$ is finite with residue field k , the framing descends to $W(k)$. There is a p -completely étale $W(k) \langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$ -algebra R_0 with $R \simeq R_0 \hat{\otimes}_{W(k)} \mathcal{O}_K$, compatibly with the framing. Indeed, for a uniformizer π of $\mathcal{O}_K$, lift the étale algebra R/π to R_0 over the corresponding $W(k)$ -algebra.

Lemma 3.4. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$. Then $(X, \mathcal{M})$ admits a finite strict étale cover by framed log affines over $\mathcal{O}_K$.*

Proof. This is surely well-known, but we include a proof for completeness. We first observe that our normal crossings setting, working strict étale locally immediately allows us to separate the local branches of D and reduce to the situation where D is a simple normal crossings divisor.

Once we are in this situation, let us zoom in around a closed point $x \in X$ (which we equip with its pullback log structure) and denote the irreducible components of D passing through x by $D_1, \dots, D_r$. We can pick a small affine Zariski neighborhood $\mathrm{Spf} R$ of x , so no other components besides these D_i meet this neighborhood.

In this situation we must now produce a framing for $\mathrm{Spf} R$. Since D is a simple normal crossings divisor after possibly shrinking further there are elements $t_1, \dots, t_r, s_{r+1}, \dots, s_n \in R$ such that $D_i = V(t_i)$ and

$$dt_1, \dots, dt_r, ds_{r+1}, \dots, ds_n$$

is a basis of $\hat{\Omega}_{R/\mathcal{O}_K}^1$. Choose constants $c_j \in \mathcal{O}_K$ such that $u_j = s_j + c_j$ has nonzero image in $\kappa(x)$; thus after Zariski localizing further, each u_j is a unit. We can even choose c_j so they are zero or one. Using the functions t_i and u_j for the images of the coordinates in $\mathbb{T}_r^n$ they define a morphism

$$\mathrm{Spf} R \xrightarrow{\square} \mathbb{T}_r^n.$$

That is, it is induced by the ring map sending T_i to t_i for $i \leq r$ and T_j to u_j for $j > r$ (the latter are invertible, which is why we localized to make u_j units).

The pullback of $V(T_1 \cdots T_r)$ is $V(t_1 \cdots t_r) = D \cap \mathrm{Spf} R$, hence $\square$ is strict as a morphism of log formal schemes. Since on stalks by construction the pullback induced by $\square$ on Ω^1 is an isomorphism this shows that the underlying morphism is p -completely étale.

Quasicompactness gives a finite cover. □

Now we can explain which results we will show for Laurent crystals on a framed log affine. The main theme will be that we want to understand exactly how they relate to Γ modules for $\Gamma = (1 + p\mathbf{Z}_p) \times \mathbf{Z}_p(1)^n$. This group Γ arises from evaluating a crystal over $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$ on the

logarithmic q -de Rham prism attached to the framing $\square$, which we now explain the construction of.

Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine with framing $\square$ over $W(k)[\zeta_p]$. Let R_0 be the $W(k)$ -algebra explained in Remark 3.3, and put $A^+ = R_0^\square[[q-1]] := R_0[[q-1]]$ and $A = R_0^\square((q-1))_p^\wedge := (A^+[1/[p]_q])_p^\wedge$. The decoration ' $\square$ ' indicates the data of the framing, which is needed to give this a log prism structure. Indeed for $\mathbb{T}_r^n$ we can define

$$W(k)\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle[[q-1]]$$

with log structure given by the chart $\mathbb{N}^r$ with basis elements $e_i \mapsto T_i$ and $\delta_{\log}$ -structure induced by the Frobenius sending $q \mapsto q^p$ and $T_i \mapsto T_i^p$. Then to generalize to a framed log affine, we lift both of these along the framing $\square$ to induce a log structure and $\delta_{\log}$ structure. Note $A^+[1/[p]_q] \simeq R$, not R_0 .

Put $\Gamma = (1+p\mathbb{Z}_p) \ltimes \mathbb{Z}_p(1)^n$. For $\gamma = (a, b_1, \dots, b_n) \in \Gamma$, regard b_i as an element of $\mathbb{Z}_p$ through the identification $\mathbb{Z}_p(1) \simeq \mathbb{Z}_p$. The formulas $q \mapsto q^a$ and $T_i \mapsto q^{pb_i}T_i$ for $i \leq n$ reduce to the identity modulo $[p]_q$. They therefore lift uniquely along the framing $\square$ to log prism automorphisms over $(\mathrm{Spf} R, \mathcal{M})$; see Lemma A.17. Thus the crystal property gives an action of Γ on every evaluation in $\mathrm{Coh}(A)$ for a coherent Laurent crystal.

More generally, in Definition A.11 we construct a category $D_\bullet((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ of solid Laurent crystals. We will need to work with this larger category of crystals, and our first result will be showing in Proposition 3.13 that evaluation $\rho_A^{\Gamma,*}$ on the log q de Rham prism induces a fully faithful functor

$$\rho_A^{\Gamma,*} : D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_\bullet(A)^{h\Gamma}.$$

By the target, we will always mean the totalization $\lim_{[n] \in \Delta} D_\bullet(C(\Gamma^n, A))$. Under a nuclearity hypothesis we will show this can be reasonably interpreted as a category of continuous semilinear Γ -modules.

We also establish a more down to earth result about roots. By a root, we mean a finite semilinear Γ -module or coherent Laurent crystals with an injective map $\mathcal{E} \rightarrow \varphi^*\mathcal{E}$. This will be crucial in giving a workable local description of finitely generated unit crystals, as the finitely generated unit condition will have a characterization in terms of roots. In particular we prove the following theorem:

Theorem 3.5. *Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine and put $\bar{A} = A/p$. Let M be a finite $\bar{A}$ -module equipped with a continuous semilinear Γ -action as above. Suppose there is a Γ -equivariant injection*

$$M \hookrightarrow \varphi^*M.$$

Then there is a coherent Laurent crystal $\mathcal{E}$ killed by p , with $\rho_A^(\mathcal{E}) = M$, and an injection $\mathcal{E} \hookrightarrow \varphi^*\mathcal{E}$ whose evaluation is the given map.*

3.1. Laurent crystals and Γ -modules. Fix a framed log affine $(\mathrm{Spf} R, \mathcal{M})$ over $\mathrm{Spf} W(k)[\zeta_p]$. We let $A^+ = R_0^\square[[q-1]]$ denote the q -de Rham prism attached to the framing $\square$. In this subsection we will define a functor

$$\rho_A^{\Gamma,*} : D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_\bullet(A)^{h\Gamma}$$

and check that it is fully faithful. As before, $(-)^{h\Gamma}$ denotes the totalization of the continuous cobar construction.

We explain some of the content of Appendix A.3 here, as we will be working in this formalism. Throughout we work with analytic rings as defined by Clausen and Scholze in [CS23] – see also [RC24] and [Kip26b] for more background material. If $(B, (d))$ is a prism, we give $B[1/d]_p^\wedge$ the stack-level p -complete analytic structure defined by the tower of Tate-Huber pairs $((B//p^n)[1/d], B//p^n)$ from [Kip26a, Example 2.1.5 and Remark 2.1.7]. For dealing with semilinear Γ -modules we will need $C(\Gamma, -)$ as well, with its natural induced analytic ring structure (as a topological ring one uses the compact-open topology). The precise definition of $D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ is given in Definition A.11 using these analytic structures, as a limit over all log prisms A in $(\mathrm{Spf} R, \mathcal{M})_\Delta^{\mathrm{op}}$.

We will show that actually there is a natural stacky interpretation of this functor, using the results in Appendix A.1 and Appendix A.3. For the reader not wishing to wade through the technical details needed to properly set up this theory, the essential point is that $(\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}$ is an analytic stack such that $D_\bullet((\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}) \simeq D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$. We postpone explaining details about this Laurent stack until §6.1 where it is explained in detail as we will only use relatively formal properties of it here. We will mostly use some properties of this stack established in the appendix, citing when necessary.

It is possible to carry the full faithfulness argument out entirely site-theoretically (indeed this was the first proof the author obtained) but the strategy is far easier to understand at a high level from a stacky perspective. We will realize $\rho_A^{\Gamma, *}$ as a pullback functor along

$$\rho_A^\Gamma : [\mathrm{AnSpec} A/\Gamma] \longrightarrow (\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}.$$

The stacky interpretation then gives a natural strategy to attack the problem. In such a situation to verify full faithfulness what one needs to do is to check that the morphism of stacks is cohomologically proper, and moreover that the canonical map $\mathcal{O} \rightarrow R\rho_{A,*}^\Gamma \mathcal{O}$ is an isomorphism. Once this is established full faithfulness is a purely formal consequence of the projection formula.

The content of the argument is then a calculation of $R\rho_{A,*}^\Gamma \mathcal{O}$, which we reinterpret as a cohomology calculation in Lemma 3.12. For this we will need an explicit description of the Čech nerve of A and how it sits inside the algebra of continuous functions $C(\Gamma^n, A)$ (see Remark 3.11 for a more conceptual description). The following lemma computes $A^{+, (1)}$ as a log prismatic envelope, which we use to obtain this description.

Lemma 3.6. *Let $A^+ = R_0^\square[[q-1]]$, and let $(A^+)^{(1)}$ be the ring of the self-coproduct of $(A^+, [p]_q, \mathcal{M})^a$ in the category of log prisms $(\mathrm{Spf} R, \mathcal{M})_\Delta^{\mathrm{op}}$. Write q_0, q_1 for the two copies of q , put $u_0 = q_1/q_0$, and let $u_i = T_{i,1}/T_{i,0}$ for $1 \leq i \leq n$. Then*

$$(A^+)^{(1)} \simeq R_0^\square[[q_0-1]][u_0^{\pm 1}, u_1^{\pm 1}, \dots, u_n^{\pm 1}] \left\{ \frac{u_0-1}{[p]_{q_0}}, \dots, \frac{u_n-1}{[p]_{q_0}} \right\}_{\delta, (p, [p]_{q_0})}^\wedge.$$

Here $\delta(u_j) = 0$ for $0 \leq j \leq n$.

Proof. This can be checked directly by matching the universal properties of the prismatic envelope. We omit this here, since we prove a more general claim in Lemma A.5. There we handle the general Breuil-Kisin case. The δ -structure used in that argument is different, but the argument is identical. $\square$

Our next task will be to understand the higher self coproducts $A^{(s)} := A^{+, (s)}[1/[p]_q]_p^\wedge$ explicitly and enhance the evaluation functor ρ_A^* for the log q -de Rham prism to keep track of the Γ action coming from a Laurent crystal structure. That is, we will show evaluation on the log q -de Rham prism induces a functor

$$\rho_A^{\Gamma, *} : D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_\bullet(A)^{h\Gamma}.$$

The crystal property does give an abstract Γ action, but does not obviously ensure continuity! Lemma A.17 writes the source $D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ as the limit

$$\lim \left(D_\bullet(A) \rightrightarrows D_\bullet(A^{(1)}) \rightrightarrows D_\bullet(A^{(2)}) \rightrightarrows \cdots \right)$$

The target of our desired functor $\rho_A^{\Gamma, *}$ can be thought of as solid A -modules with continuous semilinear Γ -action, although Lemma 3.9 tells us this is only reasonable under a nuclearity hypothesis (which will be satisfied in all applications). We will see the Γ -action on $A^+ = R_0^\square[[q-1]]$, after p -completely inverting $[p]_q$ induces maps

$$\theta_s : A^{(s)} \longrightarrow C(\Gamma^s, A).$$

A priori these land only in all functions, but we will show they land in continuous functions by realizing the image under the injection θ_s of $A^{(s)} := A^{+, (s)}[1/[p]_q]_p^\wedge$ explicitly as a subring of $C_q(\Gamma^s, A) \subset C(\Gamma^s, A)$. This then induces $\rho_A^{\Gamma, *}$ as

$$D_\bullet((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \xrightarrow{\rho_A^{\Gamma, *}} \lim_{[s] \in \Delta} D_\bullet(C(\Gamma^s, A)) =: D_\bullet(A)^{h\Gamma}.$$

We now explain the definition of $C_q(\Gamma^s, A) \subset C(\Gamma^s, A)$ by giving an explicit basis.

Definition 3.7. We use slightly different coordinates for Γ , where the coordinate x_0 for $a \in 1+p\mathbf{Z}_p$ records $x_0 = \frac{a-1}{p}$ and for $b_i \in \mathbf{Z}_p(1) \simeq \mathbf{Z}_p$ we use the coordinate x_i for $1 \leq i \leq n$. Thus we obtain a system of coordinates $x = (x_0, \dots, x_n)$ for Γ .

Set $\eta_{0,m}(x_0) = q^{m+p\binom{m}{2}}(q-1)^m \binom{x_0}{m}_{q^p}$ and $\eta_{i,m}(x_i) = q^{p\binom{m}{2}}(q-1)^m \binom{x_i}{m}_{q^p}$ for $i > 0$. Write η_α for the products of these functions in the $(n+1)$ variables and set

$$C_q(\Gamma, A) = \left(\bigoplus_{\alpha \in \mathbf{N}^{n+1}} A^+ \eta_\alpha \right)_{(p, q-1)}^\wedge [1/[p]_q]_p^\wedge.$$

We define $C_q(\Gamma^s, A)$ similarly, over indices $\alpha \in \mathbf{N}^{s(n+1)}$.²

The formula for η_α in Definition 3.7 shows that these functions are uniformly bounded. Consequently, an expansion whose coefficients tend to zero $(p, q-1)$ -adically converges uniformly, and we obtain a continuous map $C_q(\Gamma^s, A) \rightarrow C(\Gamma^s, A)$, where the target carries the compact-open topology. Here we give $C_q(\Gamma^s, A)$ the natural topology induced by the completion in Definition

²There is also an alternative formula which will be of use. Consider the divided powers

$$[m]_{q^p}! \gamma_{m, q^p}(x) = \prod_{j=0}^{m-1} ((q^p)^x - q^{pj}),$$

where $[m]_{q^p}! = \prod_{j=1}^m [j]_{q^p}$. Then $\eta_{0,m}(x_0) = q^m [p]_q^{-m} \gamma_{m, q^p}(x_0)$ and $\eta_{i,m}(x_i) = [p]_q^{-m} \gamma_{m, q^p}(x_i)$ for $i > 0$.

3.7, followed by Laurentization and p -adic completion. Lemma 3.8 shows that this map is injective, but we warn the topology on the source is not the subspace topology.

In what follows we give $A^{(s)}$ the analytic ring structure induced from the log prism $A^{+, (s)}$ in the logarithmic Čech nerve, using $\lim_n ((A^{+, (s)}/p^n)[1/d], (A^{+, (s)}/p^n))$ with their Huber pair analytic ring structures (as in the convention set in the discussion after Lemma 3.6). For $C_q(\Gamma^s, A)$ we make a similar definition, using the completed direct sum presentation as we did for its topological ring structure.

Lemma 3.8. *The map θ_s identifies $A^{(s)}$ with the subring $C_q(\Gamma^s, A) \subset C(\Gamma^s, A)$:*

$$A^{(s)} \xrightarrow{\sim} C_q(\Gamma^s, A).$$

This is further an isomorphism of analytic rings.

Proof. As above, we use $A = R_0^\square((q-1))_p^\wedge$. We first treat $s = 1$. We actually will show a stronger claim, which is that there is a natural isomorphism of adic rings

$$\theta_1 : A^{+, (1)} \simeq \left(\bigoplus_{\alpha \in \mathbb{N}^{n+1}} A^+ \eta_\alpha \right)_{(p, q-1)}^\wedge.$$

After p -completely inverting $[p]_q$ or equivalently $q-1$, this isomorphism of ordinary adic rings implies the desired isomorphism of analytic rings (as in this setting passing from topological rings to analytic rings is fully faithful).

We do this by using the explicit log prismatic envelope presentation of $A^{+, (1)}$. Recall $(A^+)^{(1)}$ from Lemma 3.6 is the $(p, q-1)$ -adic (equivalently $(p, [p]_q)$) completion of

$$A^+[u_0^{\pm 1}, \dots, u_n^{\pm 1}] \left\{ \frac{u_0 - 1}{[p]_{q_0}}, \dots, \frac{u_n - 1}{[p]_{q_0}} \right\}_\delta.$$

We used $u_0 = q_1/q_0$ and $u_i = T_{i,1}/T_{i,0}$ as shorthand. The map θ_1 can be made explicit integrally. For $g \in \Gamma$, using the coordinates $g = (x_0, \dots, x_n)$ of Definition 3.7, the evaluation of $\theta_1(a) \in \text{Map}(\Gamma, A^+)$ ³ for $a \in A^{+, (1)}$ at g is obtained by applying the A^+ -algebra map determined by $u_i \mapsto (q^p)^{x_i(g)}$ for $0 \leq i \leq n$.

Using [GLSQ26, Theorem 4.4], $A^+[u_0^{\pm 1}] \left\{ \frac{u_0 - 1}{[p]_{q_0}} \right\}_\delta^\wedge$ is, up to a change of variables, the $(p, q-1)$ -completion of the free A^+ -module on their Δ -divided powers $\omega_{0, \Delta}^{\{n\}}$ (see [GLSQ26, Definition 1.7]) as a $(p, q-1)$ -adic ring. Indeed from the discussion preceding Theorem 4.4 in [GLSQ26] they identify $\widehat{\bigoplus_{n \geq 0} A^+ \omega_{0, \Delta}^{\{n\}}}$ with the prismatic envelope of the δ -pair $(A^+[[\xi]], (\xi))$, which checking universal properties is indeed our ring (one has $\xi \mapsto q_0(u_0 - 1)$).

Now we see where θ_1 sends this basis. Using [GLSQ26, Remark 1.8(5)] provides a “blowing up” map Bl , yielding

$$A^+ \langle \xi \rangle_{q_0^p} \xrightarrow{\text{Bl}} \widehat{\bigoplus_{n \geq 0} A^+ \omega_{0, \Delta}^{\{n\}}} \simeq A^+[u_0^{\pm 1}] \left\{ \frac{u_0 - 1}{[p]_{q_0}} \right\}_\delta^\wedge \xrightarrow{\theta_1} \text{Map}(\Gamma, A^+)$$

³We haven’t yet checked the output is a continuous function.

such that the q -divided power $\xi^{[m]_{q_0^p}}$ is sent to $[p]_{q_0}^m \omega_{0,\Delta}^{\{m\}}$ under Bl . This is helpful, because we know where $\xi^{[m]_{q_0^p}}$ is sent to under the composite map $\theta_1 \circ \text{Bl}$: ξ is sent by θ_1 to

$$q((q^p)^{x_0} - 1) \in \text{Map}(\Gamma, A^+).$$

Thus we may deduce

$$[p]_{q_0}^m \theta_1(\omega_{0,\Delta}^{\{m\}}) = q^m \gamma_{m,q^p}(x_0),$$

so in particular using the alternative formula in the footnote for Definition 3.7 we see $\theta_1(\omega_{0,\Delta}^{\{m\}}) = \eta_{0,m}$ using $[p]_{q_0}$ -torsionfreeness.

Next, we handle the remaining factors corresponding to u_i for $i > 0$. The argument is more or less the same. Using [GLSQ22, Definition 1.6 and Corollary 5.6], the corresponding factor $A^+[u_i^{\pm 1}] \left\{ \frac{u_i - 1}{[p]_{q_0}} \right\}_\delta^\wedge$ is the completion of the free A^+ -module on divided powers $\omega_{i,\Delta}^{\{m\}}$, in their notation. The analogous map in [GLSQ22, Proposition 1.7] lets us deduce the formula

$$[p]_q^m \theta_1(\omega_{i,\Delta}^{\{m\}}) = \gamma_{m,q^p}(x_i).$$

Similarly we find $\theta_1(\omega_{i,\Delta}^{\{m\}}) = \eta_{i,m}$. Thus θ_1 induces a surjection onto

$$\left(\bigoplus_{\alpha \in \mathbf{N}^{n+1}} A^+ \eta_\alpha \right)_{(p,q-1)}^\wedge.$$

It is injective: evaluation at nonnegative integral values of each x_i is triangular, with $\eta_{i,m}(j) = 0$ for $m > j$ and $\eta_{i,m}(m)$ a unit times $(q-1)^m$. Only finitely many terms contribute at each such value, and A^+ is $(q-1)$ -torsion-free. Hence the integral map is, as desired, an isomorphism onto $(\bigoplus_\alpha A^+ \eta_\alpha)_{(p,q-1)}^\wedge$; the isomorphism is of $(p, q-1)$ -adic rings, so the claim about analytic ring structures also follows. One sees θ_1 lands in $C(\Gamma, A)$ by the discussion after Definition 3.7.

For general s , $A^{+, (s)} = (A^{+(1)})^{\widehat{\otimes}_{A^+}^s}$. Using this one may check

$$(A^+)^{(s)} \simeq \left(\bigoplus_{\alpha \in \mathbf{N}^{s(n+1)}} A^+ \eta_\alpha \right)_{(p,q-1)}^\wedge,$$

so by the same argument the general claim follows. $\square$

Thus we can now produce a well-defined functor

$$\rho_A^{\Gamma,*} : D_\bullet((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_\bullet(A)^{h\Gamma}.$$

We will be interested in proving this functor is fully faithful, so it is important to be able to understand the RHom in the target category as group cohomology. It seems this is only reasonable under mild hypotheses, as we explain in the following lemma.

We first fix some notation. Let G be a profinite group, and let B be a derived p -complete analytic ring with a continuous action of G (such as $A = R_0^\square((q-1))_p^\wedge$ with its Γ action). Following [Mik26, Definition 1.46], write $\text{Rep}_{B_\bullet^t}(G)$ for the category of solid modules over the twisted group algebra $B_\bullet^t[G]$; equivalently, this is the category of solid B -modules with continuous semilinear G -action. Here $\mathbf{Z}_\bullet[G]$ denotes the solid group algebra. We set

$$\text{R}\Gamma_\bullet(G, M) := \text{RHom}_{\mathbf{Z}_\bullet[G]}(\mathbf{Z}_\bullet, M) \simeq \text{RHom}_{B_\bullet^t[G]}(B, M)$$

to denote solid group cohomology. For $M \in D_{\blacksquare}(B)$, we write

$$C(G, M) := \underline{\mathrm{RHom}}_B(B[\underline{G}], M).$$

Lemma 3.9. *Let B and G be as above. Give $C(G^s, B)$ its natural analytic ring structure, and assume $C(G^s, B)$ is p -completely nuclear over B . Then the functor*

$$\varprojlim_{[s] \in \Delta} D_{\blacksquare}(C(G^s, B)) \longrightarrow \mathrm{Rep}_{B_{\blacksquare}^t}(G)$$

restricts to an equivalence from objects whose evaluation at $[0]$ is p -completely nuclear⁴ to representations whose underlying B -module is p -completely nuclear.

If $M \in D_{\blacksquare}(B)$ is the underlying solid B -module under this equivalence,

$$\varprojlim_{[s] \in \Delta} C(G^s, B) \otimes_B^{L, \blacksquare} M \simeq \varprojlim_{[s] \in \Delta} C(G^s, M) \simeq \mathrm{R}\Gamma_{\blacksquare}(G, M).$$

Recall that $D_{\blacksquare}(B)^{\mathrm{nuc}}$ consists of the objects M for which, for every compact $P \in D_{\blacksquare}(B)^{\omega}$, the canonical map

$$P^{\vee} \otimes_B^{L, \blacksquare} M \longrightarrow \underline{\mathrm{RHom}}_B(P, M)$$

is an equivalence. For a discrete analytic ring this condition is automatic for discrete modules; for example, $D_{\blacksquare}(\mathbf{Z}_{\mathrm{disc}})^{\mathrm{nuc}} = D(\mathbf{Z})$.

Proof. We use a similar argument to [Mik26, Corollary 1.50]. The Barr-Beck-Lurie theorem identifies the source and target with coalgebras in $D_{\blacksquare}(B)$ for the following comonads. The first is induced by the endofunctor

$$\mathbb{T} : D_{\blacksquare}(B) \xrightarrow{C(G, B) \otimes_B^{L, \blacksquare} -} D_{\blacksquare}(C(G, B)) \longrightarrow D_{\blacksquare}(B).$$

Here the first morphism uses the map $B \rightarrow C(G, B)$ sending $b \in B$ to the constant function (valued at b). The second morphism is pushforward along the action map $\mathrm{act} : B \rightarrow C(G, B)$ (i.e. b is sent to the function sending g to $g(b)$). The comonad structure is the obvious one (the comultiplication is induced by the multiplication on G).

The second comonad is induced by the endofunctor $\mathbb{S} : D_{\blacksquare}(B) \rightarrow D_{\blacksquare}(B)$ given by

$$\mathbb{S}(M) = C(G, M),$$

where $(bf)(g) = g(b)f(g)$ on $\mathbb{S}(M)$. Thus

$$\varprojlim_{[s] \in \Delta} D_{\blacksquare}(C(G^s, B)) \simeq \mathrm{CoAlg}_{\mathbb{T}}(D_{\blacksquare}(B)).$$

Similarly, $\mathrm{Rep}_{B_{\blacksquare}^t}(G) \simeq \mathrm{CoAlg}_{\mathbb{S}}(D_{\blacksquare}(B))$.

The map $f \otimes m \mapsto (g \mapsto f(g)m)$ defines a natural transformation $\mathbb{T} \rightarrow \mathbb{S}$ compatible with the counits and comultiplications, hence a map of comonads.

⁴By this we mean it is derived p -complete and satisfies the nuclearity property for derived p -complete objects; equivalently, it is derived p -complete and nuclear mod p .

Suppose now that M is p -completely nuclear. For the compact object $P = B[\underline{G}]$, nuclearity and the identifications $P^\vee \simeq C(G, B)$ and $\underline{R}\mathrm{Hom}_B(P, M) \simeq C(G, M)$ give

$$\mathbb{T}(M) = C(G, B) \otimes_B^{L, \blacksquare} M \simeq P^\vee \otimes_B^{L, \blacksquare} M \simeq \underline{R}\mathrm{Hom}_B(P, M) \simeq C(G, M) = \mathbb{S}(M).$$

By the assumed p -complete nuclearity of $C(G, B)$ and closure of nuclear objects under tensor products, both comonads restrict to the full subcategory of p -completely nuclear modules in $D_\blacksquare(B)$. The preceding calculation then identifies their restrictions as comonads, so coalgebra categories are therefore equivalent which proves the claim.

The cohomological comparison follows by taking $\underline{R}\mathrm{Hom}$ with the unit. $\square$

This lemma will in particular apply to $D_\blacksquare(A)^{h\Gamma}$ where $A = R_0^\square((q-1))_p^\wedge$ arises from the log q -de Rham prism for a framed log affine.

For our specific group Γ , solid group cohomology is fortunately not very complicated. We can use the resolution appearing in [Bos23, Proposition B.3 and Lemma B.4]. If $G \simeq \mathbf{Z}_p^r$, let Λ_G be the solid algebra associated with $\mathbf{Z}_p[[G]]$. For topological generators $\gamma_1, \dots, \gamma_r$, there is a finite projective resolution in solid Λ_G -modules

$$(1) \quad 0 \longrightarrow \Lambda_G^{\binom{r}{1}} \longrightarrow \dots \longrightarrow \Lambda_G^{\binom{r}{1}} \xrightarrow{(\gamma_1-1, \dots, \gamma_r-1)} \Lambda_G \longrightarrow (\mathbf{Z}_p)_\blacksquare \longrightarrow 0.$$

The proof of full faithfulness of $\rho_A^{\Gamma, *}$ will be a reduction to a cohomology computation, where we will use this resolution.

One can carry out the needed cohomology calculation in the $\eta_{i,m}$ basis of Definition 3.7, but it is actually slightly easier to work modulo p where a more natural characterization is available. Via the coordinates $x = (x_0, \dots, x_n) : \Gamma \rightarrow \mathbf{Z}_p^{n+1}$ of Definition 3.7⁵, a continuous function $f : \Gamma \rightarrow M$ with values in a complete separated module has a unique *Mahler expansion* $f(x) = \sum_\alpha a_\alpha \binom{x}{\alpha}$. Its Mahler coefficient of degree α is $a_\alpha = (\Delta_0^{\alpha_0} \dots \Delta_n^{\alpha_n} f)(0)$, where $\Delta_i f(x) = f(x + e_i) - f(x)$ and e_i is the i th standard basis vector of $\mathbf{Z}_p^{n+1}$. We will use this in the following to give an alternative characterization of C_q using Mahler expansions, at least modulo p , that we use in Lemma 3.12.

Lemma 3.10. *Let $\bar{A} = A/p$ (similarly for $\bar{A}^+$) and consider $f \in C(\Gamma, \bar{A})$. We take its Mahler expansion*

$$f(x) = \sum_{\alpha \in \mathbf{N}^{n+1}} a_\alpha \binom{x}{\alpha},$$

where $\binom{x}{\alpha} = \prod_i \binom{x_i}{\alpha_i}$. Suppose there are constants $\sigma > 1$ and C for which

$$a_\alpha \in (q-1)^{\lceil \sigma|\alpha| - C \rceil} \bar{A}^+$$

for every multi-index α . Then $f \in C_q(\Gamma, \bar{A})$.

Proof. We will prove the stronger assertion

$$\left(\bigoplus_{\alpha \in \mathbf{N}^{n+1}} \bar{A}^+ \eta_\alpha \right)_{(q-1)}^\wedge = \left(\bigoplus_{\alpha \in \mathbf{N}^{n+1}} \bar{A}^+ (q-1)^{|\alpha|} \binom{x}{\alpha} \right)_{(q-1)}^\wedge.$$

⁵The map x is only a homeomorphism, and does not respect the group structures.

We first compute the Mahler expansions of the η_α from Definition 3.7. As discussed above, to compute the coefficients if $\alpha = (\alpha_0, \dots, \alpha_n)$ we must work out the coefficients

$$(\Delta_0^{\beta_0} \dots \Delta_n^{\beta_n} \eta_\alpha)(0)$$

of the expansion of η_α into binomial coefficients.

Recall that for $i > 0$ we have $\eta_{i,m}(x_i) = (q-1)^m \cdot q^{p\binom{m}{2}} \binom{x_i}{m}_{q^p}$. Because we work in $A = R_0^\square((q-1))_p^\wedge$ the leading factor can be ignored. The Mahler expansion of a q -binomial coefficient can be computed using the recurrence

$$\Delta_i \binom{x_i}{m}_{q^p} = \binom{x_i+1}{m}_{q^p} - \binom{x_i}{m}_{q^p} = (q^{pm} - 1) \binom{x_i}{m}_{q^p} + \binom{x_i}{m-1}_{q^p}.$$

Bringing back the scaling factors $(q-1)^m \cdot q^{p\binom{m}{2}}$, this yields $\Delta_i \eta_{i,m} = (q^{pm} - 1) \eta_{i,m} + (q-1) q^{p\binom{m-1}{2}} \eta_{i,m-1}$. For $i = 0$ the result is similar, since these differ only by a factor of q^m (which is irrelevant in the argument that follows).

This immediately tells us some of the Mahler coefficients. Namely, it tells us that the coefficient of $\binom{x_i}{j}$ for $\eta_{i,m}$ vanishes for $j < m$ and is $(q-1)^m \cdot \text{unit}$ for $j = m$. The first claim is simply because $\eta_{i,m}(j) = 0$ when $j < m$. For the second, note from the previous recurrence

$$\Delta_i^m \eta_{i,m}(0) = (q-1) q^{p\binom{m-1}{2}} \Delta_i^{m-1} \eta_{i,m-1}(0)$$

so thus upon iterating this we get $(q-1)^m \cdot \text{unit}$ as the coefficient of $\binom{x_i}{m}$. For higher terms, all that matters is that a similar argument shows when $j > m$ the coefficient of $\binom{x_i}{j}$ in $\eta_{i,m}$ is in $(q-1)^{m+p(j-m)} \cdot (R_0^\square[q-1]/p)$.

In summary, we have shown that in $\overline{A}^+ = (R_0^\square[q-1]/p)$ that

$$\eta_{i,m} = u_{i,m} (q-1)^m \binom{x_i}{m} + \sum_{j>m} c_{i,m,j} (q-1)^j \binom{x_i}{j}.$$

Here $u_{i,m} \in (\overline{A}^+)^\times$, and $c_{i,m,j} \in (q-1)^{(p-1)(j-m)} \overline{A}^+$, relating the two proposed bases of $C_q(\Gamma, \overline{A})$. Since this is an upper triangular change of basis where the off diagonal part is a topologically nilpotent operator, the claim follows. More precisely this shows the $(q-1)^{|\alpha|} \binom{x}{\alpha}$ and the η_α bases are related by an operator $D(1+N)$ where D is diagonal and invertible and

$$N^r \left(\bigoplus_{\alpha \in \mathbb{N}^{n+1}} \overline{A}^+ (q-1)^{|\alpha|} \binom{x}{\alpha} \right)_{(q-1)}^\wedge \subset (q-1)^{(p-1)r} \left(\bigoplus_{\alpha \in \mathbb{N}^{n+1}} \overline{A}^+ (q-1)^{|\alpha|} \binom{x}{\alpha} \right)_{(q-1)}^\wedge,$$

hence N is topologically nilpotent. Thus we deduce the claimed isomorphism. $\square$

Remark 3.11. Interestingly this shows that modulo p we have $C_q(\Gamma, \overline{A}) = C_{\lambda-\text{an}}(\Gamma, \overline{A})$ where $\lambda = -\log_p(p-1)$ as in [Por25b]. Strictly speaking Porat does not allow $\lambda < 0$, but as noted in [Hau26, Definition 4.1, footnote 1] the definition still makes sense.

We now carry out the needed cohomology calculation.

Lemma 3.12. *Right translation defines a continuous Γ -action on $C_q(\Gamma, A)$ by $(\gamma f)(g) = f(g\gamma)$. Then there is a natural equivalence*

$$A \xrightarrow{\sim} R\Gamma_{\blacksquare}(\Gamma, C_q(\Gamma, A)).$$

Proof. By derived Nakayama it is enough to work modulo p . The proof of Lemma 3.10 shows there is a lattice

$$\left(\bigoplus_{\alpha \in \mathbb{N}^{n+1}} \overline{A}^+(q-1)^{|\alpha|} \binom{x}{\alpha} \right)_{q-1}^{\wedge} \subset C_q(\Gamma, \overline{A})$$

Let $\text{Fil}^\bullet$ be the $q-1$ -adic filtration on this lattice; this filtration is complete and separated.

Let $\gamma_0, \dots, \gamma_n$ be the usual generators of Γ , in the coordinates x_i of Γ chosen in Lemma 3.10. Writing down the action $(\gamma f)(g) := f(g\gamma)$ on $C_q(\Gamma, A)$ explicitly, we see right multiplication by γ_i acts on $g = (x_0, \dots, x_n)$ by sending $x_i \mapsto x_i + 1 + px_0$.

The desired claim amounts to showing that modulo p the complex

$$\left[C_q(\Gamma, \overline{A}) \xrightarrow{\gamma_i - \text{id}} C_q(\Gamma, \overline{A}) \right]$$

retracts onto the functions in $C_q(\Gamma, \overline{A})$ which are independent of x_i . Thus if we iterate this, using (1) shows that we get $\overline{A}$ as $R\Gamma_{\blacksquare}(\Gamma, C_q(\Gamma, A))$; one uses Hochschild-Serre to work with Γ rather than $\mathbb{Z}_p^n$, by dealing with $\mathbb{Z}_p^n$ first and then $1 + p\mathbb{Z}_p$. Let us denote by $\pi_i : C_q(\Gamma, A) \rightarrow C_q(\Gamma, A)$ the projection onto functions independent of x_i .

This can be done explicitly. The idea is to argue that $\gamma_i - \text{id}$ is a perturbation of the finite difference operator Δ_i which acts on $C_q(\Gamma, A)$ (for the coordinate x_i). Here we use that $C_q(\Gamma, A)$ is a ring and stable under the coordinate change $x_i \mapsto x_i + 1$. For Δ_i the claim is clear, because in

$$\left[C_q(\Gamma, \overline{A}) \xrightarrow{\Delta_i} C_q(\Gamma, \overline{A}) \right]$$

the kernel consists of functions independent of x_i , and it is surjective since we can define

$$h_i \left((q-1)^{|\alpha|} \binom{x}{\alpha} \right) = (q-1)^{|\alpha|} \binom{x_i}{\alpha_i + 1} \prod_{j \neq i} \binom{x_j}{\alpha_j}$$

giving a continuous endomorphism of $C_q(\Gamma, \overline{A})$ such that $\Delta_i h_i = \text{id}$.

Now we observe that

$$(\gamma_i - \text{id}) \cdot f(x_0, \dots, x_n) = f(x_0, \dots, x_i + 1 + px_0, \dots, x_n) - f(x_0, \dots, x_n).$$

In particular, $\gamma_i - \text{id}$ is nearly the finite difference operator Δ_i except for the px_0 term. We have a commutative diagram

$$\begin{array}{ccc} C_q(\Gamma, \overline{A}) & \xrightarrow{\gamma_i - \text{id}} & C_q(\Gamma, \overline{A}) \\ \text{id} \uparrow & & (\gamma_i - \text{id})h_i \uparrow \\ C_q(\Gamma, \overline{A}) & \xrightarrow{\Delta_i} & C_q(\Gamma, \overline{A}). \end{array}$$

We claim $(\gamma_i - \text{id})h_i$ differs from id by an operator that carries Fil^r into Fil^{r+p-2} . Indeed note that

$$(\gamma_i - \text{id})h_i - \text{id} = ((\gamma_i - \text{id}) - \Delta_i)h_i.$$

Following the effect of $(\gamma_i - \text{id}) - \Delta_i$ on $\binom{x_i}{\alpha}$, we note the identity

$$((\gamma_i - \text{id}) - \Delta_i) \cdot \binom{x_i}{\alpha} = \binom{(x_i + 1) + px_0}{\alpha} - \binom{x_i + 1}{\alpha} = \sum_{\ell \geq 1} \binom{x_i + 1}{\alpha - p\ell} \binom{x_0}{\ell}$$

modulo p . In particular, using this for $(q - 1)^{|\alpha|} \binom{x}{\alpha}$ shows $(\gamma_i - \text{id}) - \Delta_i$ sends Fil^r into Fil^{r+p-1} (for the $q - 1$ -adic filtration). Inspecting the formula for h_i , we then deduce $((\gamma_i - \text{id}) - \Delta_i)h_i$ sends Fil^r into Fil^{r+p-2} . As $p > 2$, this is filtration raising and $(\gamma_i - \text{id})h_i$ has an inverse.

We conclude in the previous diagram that there is a quasi-isomorphism between the rows, so the desired claim follows. $\square$

Proposition 3.13. *Evaluation on the log q -de Rham prism attached to $A^+ = R_0^\square[[q - 1]]$ induces a fully faithful functor*

$$\rho_A^{\Gamma,*} : D_\bullet((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge) \longrightarrow D_\bullet(A)^{h\Gamma}.$$

When the evaluation on the log q -de Rham prism is p -completely nuclear, the target can be thought of as semilinear Γ -modules by the equivalence of Lemma 3.9.

Proof. Under the stack/site equivalence of Proposition 6.5, $\rho_A^{\Gamma,*}$ is induced on $D_\bullet(-)$ by pullback along

$$\rho_A^\Gamma : [\text{AnSpec } A/\Gamma] \longrightarrow (\text{Spf } R, \mathcal{M})^{\mathcal{L}, \blacksquare}.$$

Indeed, taking the presentation $\text{colim}_{[n] \in \Delta^{\text{op}}} \text{AnSpec } A^{(n)}$ of the target given by Proposition A.22 after base change to $(X, \mathcal{M})^{\mathcal{L}, \blacksquare}$ and the natural presentation $\text{colim}_{[n] \in \Delta^{\text{op}}} \text{AnSpec } C(\Gamma^n, A)$ of the source one obtains the description of the functor appearing before Lemma 3.8. Here, both presentations are obtained by taking the Čech nerve of the cover by $\text{AnSpec } A$. We note that one should not construct this map by claiming only that the map $\text{AnSpec } A \rightarrow (\text{Spf } R, \mathcal{M})^{\mathcal{L}, \blacksquare}$ obtained from the fpqc epimorphism of formal stacks obtained in Lemma A.4 is Γ -equivariant, because a priori the action might not be continuous. But once one uses presentations of both stacks and Lemma 3.8 to check the action is continuous, as in the site-theoretic construction it becomes well-defined.

The assertion reduces to cohomological properness of ρ_A^Γ in the sense of [Kip26b, Definition 2.1.3.1], which we prove in Lemma 3.14. We use the same conventions as [Kip26b], and work with the six functor formalism on analytic stacks. In our case the analytic stacks all lie over $\text{AnSpec } \mathbb{Z}_{p, \blacksquare}$ with its solid analytic ring structure.

The claim that $\rho_A^{\Gamma,*}$ is fully faithful amounts to the claim that the map

$$\text{RHom}(\mathcal{E}', \mathcal{E}) \longrightarrow \text{RHom}(\rho_A^{\Gamma,*} \mathcal{E}', \rho_A^{\Gamma,*} \mathcal{E}) \simeq \text{RHom}(\mathcal{E}', R\rho_{A,*}^\Gamma \rho_A^{\Gamma,*} \mathcal{E})$$

is an isomorphism. It therefore is equivalent to check $R\rho_{A,*}^\Gamma \rho_A^{\Gamma,*} \mathcal{E} \simeq \mathcal{E}$. As ρ_A^Γ is cohomologically proper, we have access to the projection formula and can compute

$$R\rho_{A,*}^\Gamma \rho_A^{\Gamma,*} \mathcal{E} \simeq R\rho_{A,!}^\Gamma (\mathcal{O} \otimes^L \rho_A^{\Gamma,*} \mathcal{E}) \simeq R\rho_{A,*}^\Gamma \mathcal{O} \otimes^L \mathcal{E}.$$

In particular we must show the canonical map $\mathcal{O}_{(\text{Spf } R, \mathcal{M})^{\mathcal{L}, \blacksquare}} \rightarrow R\rho_{A,*}^\Gamma \mathcal{O}_{[\text{AnSpec } A/\Gamma]}$ is an isomorphism.

This task can be simplified further. Again as ρ_A^Γ is cohomologically proper we also have proper base change. We will use the square

$$\begin{array}{ccc} [\mathrm{AnSpec} A^{(1)}/\Gamma] & \longrightarrow & [\mathrm{AnSpec} A/\Gamma] \\ \downarrow f & & \downarrow \rho_A^\Gamma \\ \mathrm{AnSpec} A & \xrightarrow{\rho_A} & (\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}. \end{array}$$

This square is Cartesian, using that $\mathrm{AnSpec} A \times_{(\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}} \mathrm{AnSpec} A = \mathrm{AnSpec} A^{(1)}$ from the Čech nerve presentation of $(\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}$ in Proposition A.22. Note that the Γ action is nontrivial in $[\mathrm{AnSpec} A^{(1)}/\Gamma]$ as it is acting by right translation. For the unit $\mathcal{O} \in D_\bullet([\mathrm{AnSpec} A/\Gamma])$ properness gives a base change isomorphism

$$\rho_A^* R\rho_{A,*}^\Gamma \mathcal{O} \xrightarrow{\sim} Rf_* \mathcal{O}.$$

Formally, showing $\mathcal{O}_{(\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}} \rightarrow R\rho_{A,*}^\Gamma \mathcal{O}_{[\mathrm{AnSpec} A/\Gamma]}$ is an isomorphism then reduces to the cohomology computation that the canonical map $A \rightarrow R\Gamma_\bullet(\Gamma, C_q(\Gamma, A))$ is an equivalence. It is here that we use that the unit is p -completely nuclear, so $Rf_* \mathcal{O}$ can be computed as solid group cohomology. This calculation was done in Lemma 3.12. $\square$

The following lemma proves the properness result we needed.

Lemma 3.14. *The map*

$$\rho_A^\Gamma : [\mathrm{AnSpec} A/\Gamma] \longrightarrow (\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}$$

constructed in Proposition 3.13 is cohomologically proper.

Proof. Proposition A.22 and base change to the Laurent stack $(X, \mathcal{M})^{\mathcal{L}, \blacksquare}$ show that $\rho_A : \mathrm{AnSpec} A \rightarrow (\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}$ is a representable proper effective epimorphism. Thus we only need to deal with the quotient by Γ .

Since properness is local on the target for such covers by [Kip26b, Remark 1.1.3.7], it suffices to check the claim for

$$[\mathrm{AnSpec} A^{(1)}/\Gamma] \longrightarrow \mathrm{AnSpec} A$$

where we again use the square

$$\begin{array}{ccc} [\mathrm{AnSpec} A^{(1)}/\Gamma] & \longrightarrow & [\mathrm{AnSpec} A/\Gamma] \\ \downarrow f & & \downarrow \rho_A^\Gamma \\ \mathrm{AnSpec} A & \xrightarrow{\rho_A} & (\mathrm{Spf} R, \mathcal{M})^{\mathcal{L}, \blacksquare}. \end{array}$$

This property is stable under base change, so this can be also checked using A^+ and $A^{+, (1)}$. The right translation action involved in $[\mathrm{AnSpec} A^{(1)}/\Gamma]$ fixes the image of A^+ , giving a factorization of the map as

$$[\mathrm{AnSpec} A^{+, (1)}/\Gamma] \longrightarrow \mathrm{AnSpec} A^+ \times B\Gamma \longrightarrow \mathrm{AnSpec} A^+.$$

After base change, pulling the first map back along $\mathrm{AnSpec} A^+ \rightarrow \mathrm{AnSpec} A^+ \times B\Gamma$ gives $\mathrm{AnSpec} A^{+,(1)} \rightarrow \mathrm{AnSpec} A^+$, which is proper by Lemma A.13. It remains to show that

$$B\Gamma \longrightarrow \mathrm{AnSpec} \mathbf{Z}_{p,\blacksquare}$$

is proper for the six functor formalism on analytic stacks. One may use an argument similar to [HM24, Proposition 5.3.2]. The map $g : \mathrm{AnSpec} \mathbf{Z}_{p,\blacksquare} \rightarrow B\Gamma$ is representable and proper, while $B\Gamma \rightarrow \mathrm{AnSpec} \mathbf{Z}_{p,\blacksquare}$ is 1-truncated and its composite with g is the identity. By [HM24, Corollary 4.7.5(iii)], it remains only to verify that $g_*\mathcal{O} = C(\Gamma, \mathbf{Z}_p)$ is descendable. The finite resolutions in (1), applied to the normal subgroup $\mathbf{Z}_p(1)^n$ and quotient $1 + p\mathbf{Z}_p \simeq \mathbf{Z}_p$, show that the trivial representation $\mathbf{Z}_p$ is perfect over $\mathbf{Z}_p[[\Gamma]]$. Taking derived solid $\mathbf{Z}_p$ -duals and using $\mathbf{Z}_p[[\Gamma]]^\vee \simeq C(\Gamma, \mathbf{Z}_p)$ gives

$$\mathbf{Z}_p \in \mathrm{Thick}(C(\Gamma, \mathbf{Z}_p)) \subset D_\blacksquare(B\Gamma),$$

as desired. $\square$

3.2. Roots in Laurent crystals. We now turn our attention to proving Theorem 3.5. As we now impose a perfectness or coherence condition, condensed mathematics is not particularly relevant in this subsection.

In Appendix A.1 we show in Corollary A.19 that perfect Laurent crystals have a canonical t -structure in the normal crossings setting, allowing us to define $\mathrm{Coh}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$.

For a framed log affine, put $A = R_0^\square((q-1))_p^\wedge$ as before. We use A to mean this throughout this subsection, and use $\bar{A} := A/p$. One can understand the category of perfect A -complexes with continuous Γ action or $\mathrm{Perf}(A)^{h\Gamma}$ as the totalization

$$(2) \quad \lim \left(\mathrm{Perf}(A) \begin{smallmatrix} \xrightarrow{p_0} \\ \xrightarrow{p_1} \end{smallmatrix} \mathrm{Perf}(C(\Gamma, A)) \begin{smallmatrix} \rightrightarrows \\ \rightrightarrows \end{smallmatrix} \mathrm{Perf}(C(\Gamma^2, A)) \begin{smallmatrix} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{smallmatrix} \cdots \right)$$

where A has the topology specified after Lemma 3.6. This is compatible with the previous very general description $\lim_{[s] \in \Delta} D_\blacksquare(C(\Gamma^s, A))$ by taking dualizable objects, which will only depend on the underlying ring (i.e. evaluation on $*$ is an equivalence with ordinary perfect complexes). These are automatically p -completely nuclear, so by Lemma 3.9 there is no subtlety about the relation to modules with semilinear Γ -action.

For coherent objects, to write down the data it is enough to keep the terms through $C(\Gamma^2, \bar{A})$ in (2). The descent data can naturally be encoded as a comodule over the Hopf algebroid $(\bar{A}, C(\Gamma, \bar{A}))$. Explicitly, an object of the three-term totalization is a finite $\bar{A}$ -module M together with an isomorphism

$$\alpha : C(\Gamma, \bar{A}) \otimes_{p_1, \bar{A}}^\blacksquare M \xrightarrow{\sim} C(\Gamma, \bar{A}) \otimes_{p_0, \bar{A}}^\blacksquare M$$

satisfying the identity and cocycle conditions. Writing $p_0(a)(g) = a$ and $p_1(a)(g) = g(a)$, the rule $\delta(m) = \alpha(1 \otimes m)$ identifies α with a p_1 -semilinear coaction $\delta : M \rightarrow C(\Gamma, \bar{A}) \otimes_{p_0, \bar{A}}^\blacksquare M$. The identity and cocycle conditions become the counit and coassociativity axioms, respectively, giving the equivalence with finite comodules over the Hopf algebroid.

Upon restricting to Coh the description is also compatible with classical theory describing Γ -modules, where one equips finite modules with their weak topology as in [SV11, Lemmas 8.2 and 8.22] (which discusses the case of a point). In particular, one may check that there is an equivalence between $\mathrm{Coh}(A)^{h\Gamma}$ (the heart of the cobar description described above) and the category of finite A -modules with a continuous Γ -action, with the A -module given its weak topology. For clarity,

over a topological ring A the weak topology on a finitely presented module M is the quotient topology induced by

$$A^n \twoheadrightarrow M$$

which is independent of the choice of generators. The following lemma clarifies the situation.

Lemma 3.15. *Let A be a topological ring, and let Γ be a profinite group acting continuously on A . We make two topological assumptions on A :*

- *The topology on A is complete Hausdorff metrizable.*
- *The ring A has the property that a finitely generated module $M \subseteq A^n$ has its weak topology agree with its subspace topology.*

Let $\text{Mod}^{\text{fp}}(A)$ be the category of finitely presented A -modules. Then there is an equivalence

$$F : \text{Mod}^{\text{fp}}(A)^{h\Gamma} \simeq \left\{ \begin{array}{l} M \in \text{Mod}_A^{\text{fp}} \text{ with } \Gamma \times M \rightarrow M \\ \text{continuous for the weak topology on } M \end{array} \right\}.$$

Proof. The main content of the argument is actually just checking F is well-defined: through the cobar description we immediately get some Γ -module M , but there is no immediate reason for the action to be continuous for the weak topology.

A priori, the Γ action could be discontinuous for the weak topology. What we must check is that the coaction map

$$M \longrightarrow C(\Gamma, A) \otimes_{p_0, A} M \simeq C(\Gamma, M)$$

is continuous for the weak topology on M . This is really two claims: that the second isomorphism is an isomorphism of topological modules where $C(\Gamma, A)$ has the compact-open topology and M has the weak topology, and that the map is continuous. Once this is established the equivalence is clear, since the inverse to F simply forgets all the topological data.

We will use the following general property of $C(\Gamma, -)$ with respect to exact sequences.

Lemma 3.16. *Let A be complete Hausdorff metrizable, and let*

$$0 \longrightarrow K \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

be an exact sequence of left modules where A^n has its natural product topology, K has the subspace topology and is closed, and M the quotient topology. Then

$$0 \longrightarrow C(\Gamma, K) \longrightarrow C(\Gamma, A^n) \longrightarrow C(\Gamma, M) \longrightarrow 0$$

is a strictly exact sequence of topological modules over $C(\Gamma, A)$ where each is equipped with the compact-open topology.

Here the term “strict exact” means that the leftmost term has the subspace topology and the rightmost term has the quotient topology.

Proof. We first check $C(\Gamma, K) \rightarrow C(\Gamma, A^n)$ is injective and acquires the subspace topology. For injectivity we may test this at the level of sets, where it is immediate since $K \rightarrow A^n$ is injective any function which is zero on A^n is necessarily zero on K . Necessarily $C(\Gamma, K)$ acquires the subspace topology, as a subbasis of opens are the collections $V_{\Gamma', U} = \{f \in C(\Gamma, K) : f(\Gamma') \subset U\}$ where $\Gamma' \subseteq \Gamma$ is compact and U open. The subspace topology on the other hand asks for the same opens but where $U \subseteq A^n$ is open and we intersect with K . Since K has the subspace topology we can match the subbases for the topology.

Middle exactness is also clear, as the functions sent to 0 under $C(\Gamma, A^n) \rightarrow C(\Gamma, M)$ are precisely those valued in K . Thus the nontrivial content is surjectivity of $C(\Gamma, A^n) \rightarrow C(\Gamma, M)$, and that the target acquires the quotient topology.

Using [Wig73, Theorem M, p. 86], every continuous $f : \Gamma \rightarrow M$ lifts to $f : \Gamma \rightarrow A^n$ viewing both as additive topological groups. This proves set-theoretic surjectivity. We must also show that $C(\Gamma, M)$ acquires the quotient topology, which is what we need for strictness. The map is continuous, so we must check $C(\Gamma, A^n) \rightarrow C(\Gamma, M)$ is an open map.

Letting $\pi : A^n \rightarrow M$ denote the quotient map, by abuse of notation we also use π for the map $C(\Gamma, A^n) \rightarrow C(\Gamma, M)$. By virtue of being a homomorphism we must check $\pi_*(V_{\Gamma, U}) = V_{\Gamma, \pi(U)}$ (in our previous notation for the subbasis of the compact-open topology; this is a basis around zero). More precisely we must check $f : \Gamma \rightarrow \pi(U) \subset M$ lifts to $\tilde{f} : \Gamma \rightarrow U \subset A^n$. Viewing f as a map to M the previous result gives a lift, which need not land in U . Around $\gamma \in \Gamma$ we may always correct this lift in an open neighborhood of γ by taking $\tilde{f} - k_\gamma$, where $k_\gamma \in K$. Since Γ is profinite we can take a finite clopen partition into Γ_i (not as subgroups!) so Γ_i are contained in these opens, allowing us to correct $\tilde{f}$ piecewise by k_{γ_i} to land in U . This shows the claim. $\square$

Granting the lemma the proof is straightforward. When $M = A^n$ the claim that F is well-defined is clear. For general M , consider the presentation $0 \rightarrow K \rightarrow A^n \twoheadrightarrow M$. Since we assume the weak and subspace topologies on K are the same, it follows that $K \subset A^n$ is a complete additive subgroup hence closed (completeness follows from giving K the quotient topology from $A^m \twoheadrightarrow K$).

By the lemma we therefore get an exact sequence

$$0 \rightarrow C(\Gamma, K) \rightarrow C(\Gamma, A^n) \rightarrow C(\Gamma, M) \rightarrow 0$$

and applying it again with $A^m \twoheadrightarrow K$ we get a right exact sequence

$$C(\Gamma, A^m) \rightarrow C(\Gamma, A^n) \rightarrow C(\Gamma, M) \rightarrow 0.$$

For the first two we have $C(\Gamma, A) \otimes_{p_0, A} A^n \simeq C(\Gamma, A^n)$ and similarly for A^m , so keeping track of topologies we get an isomorphism of topological modules $C(\Gamma, A) \otimes_{p_0, A} M \simeq C(\Gamma, M)$ where M has its weak topology.

Finally, continuity of $M \rightarrow C(\Gamma, A) \otimes_{p_0, A} M$ is automatic from continuity of $A \rightarrow C(\Gamma, A)$. The claim then follows. $\square$

Both assumptions are easy to verify for $R_0^\square((q-1))_p^\wedge$.

Lemmas A.17 and A.16 give the same comodule description for Laurent crystals that we just discussed for semilinear Γ -modules. Put $\overline{A}^{(m)} = A^{(m)}/p$. In the coaction formulation, coherent Laurent crystals are described by replacing $C(\Gamma, \overline{A})$ and $C(\Gamma^2, \overline{A})$ with $\overline{A}^{(1)}$ and $\overline{A}^{(2)}$. For the

following lemma, we note there is a natural identification $C(\Gamma^s, N) \simeq C(\Gamma^s, \bar{A}) \otimes_{\bar{A}, p_0}^{\blacksquare} N$ for a finite $\bar{A}$ -module N equipped with its weak topology, as we showed in Lemma 3.15. We keep the solid decoration in what follows to remind the reader that we are working with modules with their weak topology or canonical solid structure as dualizable objects; in the coherent setting the distinction does not matter as they are identified after passing to the associated condensed module.

As usual let $\bar{A} = A/p$, and let $p_0 : \bar{A} \rightarrow \bar{A}^{(s)}$ be the map in the Čech nerve corresponding to the canonical inclusion $[0] \hookrightarrow [s]$. Recall the maps θ_s were introduced in Lemma 3.8 to identify $A^{(s)}$ as subrings of $C(\Gamma^s, A)$.

Lemma 3.17. *Let $s = 1$ or 2 . For every finite $\bar{A}$ -module N , the map θ_s induces an injection*

$$\bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} N \longrightarrow C(\Gamma^s, N).$$

If $M \subset N$ is a finite submodule stable under Γ^s , then inside $C(\Gamma^s, N)$ one further has

$$\bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} M = (\bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} N) \cap C(\Gamma^s, M).$$

Proof. The first claim is straightforward using the basis η_α from Definition 3.7. An element $f \in \bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} N$ has an expansion

$$f = \sum_{\alpha \in \mathbf{N}^{s(n+1)}} m_\alpha \eta_\alpha.$$

Just as in the proof of Lemma 3.10 for $\alpha, \ell \in \mathbf{N}^{s(n+1)}$ we have $\eta_\alpha(\ell) = 0$ unless $\alpha \leq \ell$ coordinate-wise, while $\eta_\ell(\ell) \in \bar{A}^\times$. If f evaluates to zero, then for the remaining terms we get

$$0 = f(\ell) = \sum_{\alpha \leq \ell} m_\alpha \eta_\alpha(\ell).$$

This sum is finite. Induction on $|\ell|$ therefore gives $m_\ell = 0$ for every ℓ , proving injectivity.

We now attack the second claim, namely that if $M \subset N$ is a finite stable under Γ^s , then

$$\bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} M = (\bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} N) \cap C(\Gamma^s, M).$$

It is useful to observe that the map $p_0 : A \rightarrow A^{(s)}$ is p -completely flat: before applying $(-)[1/[p]_q]_p^\wedge$ we see $(p, [p]_q)$ -complete flatness follows from flatness of prismatic envelopes (see [Mao24, Proposition 5.52]). Since A^+ is Noetherian, [Bha20, Lemma 5.15] makes $p_0 : A \rightarrow A^{(s)}$ p -completely flat as it is the p -completion of a flat map. Its reduction modulo p , $p_0 : \bar{A} \rightarrow \bar{A}^{(s)}$, is then flat.

Consider now the commutative square

$$\begin{array}{ccc} \bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} N & \longrightarrow & C(\Gamma^s, N) \\ \downarrow & & \downarrow \\ \bar{A}^{(s)} \otimes_{\bar{A}, p_0}^{\blacksquare} (N/M) & \longrightarrow & C(\Gamma^s, N/M). \end{array}$$

The lower horizontal map is injective by the first claim of the lemma. Flatness identifies the kernel of the left vertical map with $\overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} M$. If an element of the upper left term $\overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} N$ maps into $C(\Gamma^s, M)$, its image in the lower right term $C(\Gamma^s, N/M)$ vanishes. It therefore vanishes in the lower left term $\overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} (N/M)$ and belongs to $\overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} M$ (using injectivity of the lower horizontal map). Thus

$$(\overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} N) \cap C(\Gamma^s, M) \subseteq \overline{A}^{(s)} \otimes_{\overline{A}, p_0}^{\blacksquare} M.$$

The reverse inclusion is clear since $M \subset N$. $\square$

Lemma 3.18. *Let M be a finite $\overline{A}$ -module with continuous semilinear Γ -action. For $k \gg 0$, the coaction of $\varphi^{*k}M$ factors through*

$$C_q(\Gamma, \overline{A}) \otimes_{\overline{A}, p_0}^{\blacksquare} \varphi^{*k}M \subset C(\Gamma, \varphi^{*k}M).$$

The same holds with Γ^2 as needed for the cocycle condition.

Proof. We explain the proof for Γ , as the argument for Γ^2 is similar. In Lemma 3.10 we can immediately adapt it with coefficients to give a criterion for membership in $C_q(\Gamma, \overline{A}) \otimes_{\overline{A}, p_0}^{\blacksquare} M \subset C(\Gamma, M)$ in terms of how rapidly Mahler coefficients tend to zero. Here to make sense of the target we have equipped M with its weak topology.

For $f \in C(\Gamma, M)$, take its Mahler expansion

$$f(x) = \sum_{\alpha \in \mathbb{N}^{n+1}} m_{\alpha} \binom{x}{\alpha},$$

where $\binom{x}{\alpha} = \prod_i \binom{x_i}{\alpha_i}$. The desired criterion is then that if there are $\sigma > 1$ and C for which

$$m_{\alpha} \in (q-1)^{\lceil \sigma|\alpha| - C \rceil} M^+$$

for every multi-index α then $f \in C_q(\Gamma, M) = C_q(\Gamma, \overline{A}) \otimes_{\overline{A}, p_0}^{\blacksquare} M$. This is easy to see, as taking the existing criterion for $\overline{A}$ and choosing a finite free surjection $\overline{A}^{+\oplus n} \rightarrow M^+$ and inverting $q-1$ proves the criterion for finite coefficients.

Now we consider our M with a continuous semilinear Γ -action, and view it as a module M with a coaction

$$M \longrightarrow C(\Gamma, A) \otimes_{p_0, A} M$$

satisfying counit and coassociativity axioms. Here we are using the perspective introduced at the start of this section that one may think of such a module as a comodule over the Hopf algebroid $(\overline{A}, C(\Gamma, \overline{A}))$. In Lemma 3.15 we explained that we can also think of this as a continuous map $M \rightarrow C(\Gamma, M)$ where we equip M with the weak topology.

What we wish to show is that upon taking a Frobenius pullback we have $\varphi^{*k}M \rightarrow C(\Gamma, \varphi^{*k}M)$ factoring through $C_q(\Gamma, \overline{A}) \otimes_{\overline{A}, p_0}^{\blacksquare} \varphi^{*k}M$. Conceptually the remaining argument is very simple using the observation in Remark 3.11: one observes that such Γ modules always have a (lower) λ -analytic Γ -action for some λ , and taking Frobenius pullbacks improves the λ until the factorization holds.

Making this precise of course requires some more work. The notion of λ -analytic vectors is defined in [Por25a, Definition 2.10] but we will use the slightly different convention in [Hau26, Definition 4.1]. In this convention we put

$$C_{\lambda-\text{an}}(\Gamma, M) = \left\{ \sum_{\alpha} m_{\alpha} \binom{x}{\alpha} : \nu_{q-1}(m_{\alpha}) - \frac{|\alpha|}{p^{\lambda}(p-1)} \rightarrow \infty \right\}.$$

Smaller λ is a stronger condition. One also has upper λ -analytic vectors, replacing $\frac{|\alpha|}{p^{\lambda}(p-1)}$ with $\sum \nu_p \left(\left\lfloor \frac{\alpha_i}{p^{\lambda}} \right\rfloor! \right)$. One can check

$$C_{\lambda-\text{an}}(\Gamma, M) \subseteq C^{\lambda-\text{an}}(\Gamma, M) \subseteq C_{(\lambda+\varepsilon)-\text{an}}(\Gamma, M)$$

for every $\varepsilon > 0$ (see the similar argument in [Por25b, Lemma 4.4(2)].)

In this specific case the results in [Por25b] do not literally apply, but [Por25b, Remark 5.10] actually already contains everything needed for the argument. There Porat observes that for some small $\Gamma_0 \subset \Gamma$ that $(g-1)(q-1)^i \bar{A}^+ \subset (q-1)^{i+1} \bar{A}^+$. Picking Γ_0 small enough to be uniform and picking a coordinate $x : \mathbf{Z}_p^{n+1} \rightarrow \Gamma_0$ (we remark this is not a group homomorphism in our case) specified by generators γ_i Porat computes for $g = \prod_i \gamma_i^{x_i}$ and $x = (x_0, \dots, x_n)$ that

$$g(a) = \sum_{\alpha \in \mathbf{N}^{n+1}} \binom{x}{\alpha} (\gamma_0 - 1)^{\alpha_1} \cdots (\gamma_n - 1)^{\alpha_n} (a),$$

concluding that any element of $\bar{A}$ belongs to $C^{\lambda-\text{an}}(\Gamma_0, \bar{A})^{\Gamma_0}$ for $\lambda \gg 0$, using the fact that $(g-1)$ is $q-1$ -adic filtration raising. Of course, the same argument applies with finite coefficients M equipped with their weak topology, with the elements being generators. One must be slightly careful here to run the first step of the argument as we must pick a finite Γ -stable lattice M^+ over $\bar{A}^+$ of M , but this follows by continuity of the Γ action for the weak topology on M . Given a finite set of generators S for any lattice M^+ , we observe that by continuity $\Gamma S \subset (q-1)^{-N} M^+$ (here we also use compactness of Γ). Now we simply replace M^+ by $\bar{A}^+ \cdot \Gamma S \subset (q-1)^{-N} M^+$. Since continuity on S gives the original lattice an open stabilizer in Γ , it follows that this module is finite, and by construction it is Γ -stable. The same argument gives $(g-1)(q-1)^i M^+ \subset (q-1)^{i+1} M^+$.

Thus we obtain that, before even taking a Frobenius pullback, the coaction factors as

$$M \longrightarrow C^{\lambda-\text{an}}(\Gamma, M).$$

Note however that in Remark 3.11 we used lower locally analytic vectors. However up to modifying λ we can obtain the same claim for $C_{\lambda-\text{an}}(\Gamma, M)$, by the earlier inclusions. The desired result then follows by observing that a Frobenius pullback improves λ to $\lambda - k$ in the factorization of the coaction until Remark 3.11 applies. $\square$

Proposition 3.19. *Let M be a finite $\bar{A}$ -module with continuous semilinear Γ -action. For $k \gg 0$, the module $\varphi^{*k} M$ arises from a coherent Laurent crystal killed by p .*

Proof. Choose k large enough for Lemma 3.18 in degrees one and two, and put $N = \varphi^{*k} M$. Lemmas 3.8 and 3.17 give injective vertical maps in the commutative diagram

$$\begin{array}{ccccc}
N & \rightrightarrows & C(\Gamma, N) & \rightrightarrows & C(\Gamma^2, N) \\
\parallel & & \uparrow & & \uparrow \\
N & \rightrightarrows & \overline{A}^{(1)} \otimes_{\overline{A}, p_0}^{\blacksquare} N & \rightrightarrows & \overline{A}^{(2)} \otimes_{\overline{A}, p_0}^{\blacksquare} N
\end{array}$$

We are given a full descent datum for N on the upper row, and want to show that it actually comes from the lower row. The coaction $m \mapsto (g \mapsto g(m))$ factors uniquely through the lower degree one term $\overline{A}^{(1)} \otimes_{\overline{A}, p_0}^{\blacksquare} N$ by Lemma 3.18. Note also that the diagram commutes using the description of the vertical maps from Lemma 3.8.

The counit and coassociativity identities hold in the upper row, hence in the lower row by injectivity in degrees one and two from Lemma 3.17. Thus the lower row defines a descent datum on N as a coherent Laurent crystal and the claim follows. $\square$

Lemma 3.20. *Let $M \subset N$ be finite $\overline{A}$ -modules with continuous semilinear Γ -action. If N is the evaluation of a coherent Laurent crystal killed by p and M is Γ -stable, then M also arises from a coherent Laurent crystal.*

Proof. One can identify the crystal's coaction with

$$N \longrightarrow \overline{A}^{(1)} \otimes_{\overline{A}, p_0}^{\blacksquare} N \hookrightarrow C(\Gamma, N).$$

We are given $M \subset N$ equipped with a (coaction) map $M \rightarrow C(\Gamma, M)$, such that

$$\begin{array}{ccc}
N & \longrightarrow & C(\Gamma, N) \\
\uparrow & & \uparrow \\
M & \longrightarrow & C(\Gamma, M)
\end{array}$$

commutes. By Lemma 3.17 we see the coaction restricts to M giving

$$M \longrightarrow \overline{A}^{(1)} \otimes_{\overline{A}, p_0}^{\blacksquare} M.$$

The cocycle and counit conditions follow by a similar argument. $\square$

Proof of Theorem 3.5. Let $\beta : M \hookrightarrow \varphi^* M$ be the given map. Iterating β embeds both M and $\varphi^* M$ as Γ -stable submodules of $N = \varphi^{*k} M$ for every $k \geq 1$, as φ is flat on $\overline{A}$.

Now choose k large enough for Proposition 3.19, and let $\mathcal{F}$ be the resulting coherent crystal with evaluation N . Lemma 3.20 gives coherent subcrystals $\mathcal{E}, \varphi^* \mathcal{E} \subset \mathcal{F}$ with evaluations M and $\varphi^* M$ (full faithfulness in Proposition 3.13 shows the second crystal is really the Frobenius pullback; nuclearity is automatic for coherent crystals). Proposition 3.13 also lifts β uniquely to a morphism

$$\tilde{\beta} : \mathcal{E} \longrightarrow \varphi^* \mathcal{E}.$$

Its kernel evaluates to zero on the log q -de Rham prism, so t -exactness and conservativity from Lemma A.17 and Corollary A.19 make $\tilde{\beta}$ injective. $\square$

As promised we now investigate other bases.

Corollary 3.21. *Both Proposition 3.13 and Theorem 3.5 hold over $W(k)[\zeta_p^\ell]$ for $\ell > 0$, by changing to $\Gamma = (1 + p^\ell \mathbf{Z}_p) \ltimes \mathbf{Z}_p(1)^n$ and keeping the same base $A = R_0^\square((q-1))_p^\wedge$.*

For $\ell = 0$, again keeping the same base $A = R_0^\square((q-1))_p^\wedge$ but using $\Gamma = \mathbf{Z}_p^\times \ltimes \mathbf{Z}_p(1)^n$ Proposition 3.13 holds and Theorem 3.5 also holds.

Proof. For $\ell > 0$ the modification is straightforward, and can be done by simply setting the prismatic ideal in the q -de Rham prism to be $\varphi^{\ell-1}([p]_q)$. The toric factor acts by $T_i \mapsto q^{p^\ell b_i} T_i$, so this action is trivial modulo $\varphi^{\ell-1}([p]_q)$.

For $\ell = 0$ and p odd, one can evade redoing the calculations in Proposition 3.13 in favor of a simpler calculation. We will first show that the pullback along

$$f : [W(k)[\zeta_p]^{\mathcal{L}, \blacksquare} / \mu_{p-1}] \longrightarrow W(k)^{\mathcal{L}, \blacksquare}$$

is fully faithful on $D_\blacksquare(-)$. Here the μ_{p-1} action comes from functoriality and the Galois action on $W(k)[\zeta_p]/W(k)$. One can use the same argument as in Proposition 3.13 to reduce to checking if the pushforward of the structure sheaf on the source is $\mathcal{O}_{W(k)^{\mathcal{L}, \blacksquare}} = \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$ via the projection formula after we verify f is prim (for the six functor formalism on analytic stacks). If we know this, then it is formal to deduce the claim in the general case for $[(R_0[\zeta_p], \mathcal{M})^{\mathcal{L}, \blacksquare} / \mu_{p-1}] \rightarrow (R_0, \mathcal{M})^{\mathcal{L}, \blacksquare}$: one may use base change for prim maps to deduce the required isomorphism. Verifying f is prim turns out to be formal from our results so far. One can use the diagram

$$\begin{array}{ccc} [\mathrm{AnSpec} B / \mathbf{Z}_p^\times] & \longrightarrow & [\mathrm{AnSpec} W(k)((q-1))_p^\wedge / \mathbf{Z}_p^\times] \\ \downarrow & & \downarrow \rho_{W(k)((q-1))_p^\wedge}^\Gamma \\ & & [W(k)[\zeta_p]^{\mathcal{L}, \blacksquare} / \mu_{p-1}] \\ & & \downarrow f \\ \mathrm{AnSpec} W(k)((\tilde{p}))_p^\wedge & \longrightarrow & W(k)^{\mathcal{L}, \blacksquare} \end{array}$$

where the outer rectangle is Cartesian (with $\tilde{p}$ as in [BL22a, Notation 3.8.9]). We may compute B as a prismatic envelope using Lemma A.6; the analytic realization $(-)^{\blacksquare}$ of Definition 6.1 and passage to Laurent stacks preserve finite limits.

The upper right arrow $\rho_{W(k)((q-1))_p^\wedge}^\Gamma$ is proper, as in the argument of Proposition 3.13; moreover the argument also shows $\rho_{W(k)((q-1))_p^\wedge}^\Gamma \mathcal{O} \simeq \mathcal{O}$. The left vertical map $[\mathrm{AnSpec} B / \mathbf{Z}_p^\times] \rightarrow \mathrm{AnSpec} W(k)((\tilde{p}))_p^\wedge$ is proper by the same argument as in Proposition 3.13, using Lemma A.13 and properness of $B\mathbf{Z}_p^\times$. Hence the composite of the right vertical maps is proper, since the bottom map is an effective epimorphism of analytic stacks by Proposition A.20. Now [HM24, Corollary 4.7.5(ii)] shows that f is prim, which gives the required projection formula and base change properties for the argument by [HM24, Corollary 4.5.11(ii) and Lemma 4.5.13(ii)].

Thus we check $\mathcal{O}_{W(k)^{\mathcal{L}, \blacksquare}} \simeq Rf_* \mathcal{O}_{[W(k)[\zeta_p]^{\mathcal{L}, \blacksquare} / \mu_{p-1}]}$. One can use the same argument as in Proposition 3.13 to reduce to checking if the pushforward of the structure sheaf on the source is $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge$.

Using Proposition 6.5, we may work site-theoretically, so one may reduce to checking the stronger claim

$$W(k)[[\tilde{p}]] \xrightarrow{\sim} \Delta_{W(k)[\zeta_p]/W(k)[[\tilde{p}]]}^{h\mu_{p-1}}.$$

Here we use conservativity of evaluation on $W(k)[[\tilde{p}]]$ to make this reduction. This can be checked directly, using the Hodge-Tate comparison. Indeed, $L_{W(k)[\zeta_p]/W(k)} \simeq \hat{\Omega}_{W(k)[\zeta_p]/W(k)}^1$ compatibly with the Galois action on $W(k)[\zeta_p]/W(k)$, but the target has no ordinary μ_{p-1} invariants; higher exterior powers also follow, as in this case they are just shifts of $\hat{\Omega}_{W(k)[\zeta_p]/W(k)}^1$ with Breuil-Kisin twists. The μ_{p-1} invariants are exact so the desired result follows from

$$W(k)[[\tilde{p}]] = W(k)[[q-1]]^{\mu_{p-1}}.$$

We now explain the second assertion, the proof of Theorem 3.5. The Čech calculation carries over for the log prism $(R_0^\square[[\tilde{p}]], (\tilde{p}), \mathcal{M})$ – this prism is constructed analogously to the log q -de Rham prism via a framing, but we note that one must be cautious as it does not possess a full Γ action. Its first Čech term is

$$(R_0^\square[[\tilde{p}]])^{(1)} \simeq R_0^\square[[\tilde{p}_0, \tilde{p}_1]][u_1^{\pm 1}, \dots, u_n^{\pm 1}] \left\{ \frac{\tilde{p}_1}{\tilde{p}_0}, \frac{u_1 - 1}{\tilde{p}_0}, \dots, \frac{u_n - 1}{\tilde{p}_0} \right\}_{\delta, (p, \tilde{p}_0)}^\wedge,$$

where $u_i = T_{i,1}/T_{i,0}$ and $\delta(u_i) = 0$. As we work modulo p always, what is really needed is an analogue of Lemma 3.10. This doesn't seem to quite work, but we can use the following trick. By a similar argument we may deduce

$$((R_0^\square[[\tilde{p}]])^{(1)} \otimes_{R_0^\square[[\tilde{p}]]} R_0^\square[[q-1]])/p \simeq \left(\bigoplus_{\alpha \in \mathbf{N}^{n+1}} (R_0^\square/p)[[q-1]] \tilde{p}^{\alpha_0} (q-1)^{|\alpha_1, \dots, \alpha_n|} \binom{x}{\alpha} \right)_{(q-1)}^\wedge.$$

Indeed for the toric directions we can repeat verbatim the same argument as already appearing in Lemma 3.8 for $\eta_{i,m}$ when $i > 0$, which then works identically through the method of Lemma 3.10. For the remaining part one observes that

$$k[[\tilde{p}_0, \tilde{p}_1]]\{\tilde{p}_1/\tilde{p}_0\}_{\delta, (p, \tilde{p}_0)} \simeq \left(\bigoplus_{\alpha_0 \in \mathbf{N}} k[[\tilde{p}_0]] \tilde{p}^{\alpha_0} \binom{x}{\alpha} \right)_{\tilde{p}_0} \subset C(1 + p\mathbf{Z}_p, k[[\tilde{p}_0]]).$$

Combining these gives the claimed identification, noting that we need base extension primarily to deal with the toric part (otherwise $q-1$ is not an element of the ring, so the identification makes no sense).

This identification shows that a single Frobenius pullback makes

$$\mathcal{E} \in \text{Coh}((R_0 \otimes_{W(k)} W(k)[\zeta_p], \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{h\mu_{p-1}}$$

descend to $\text{Coh}((R_0, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$. Indeed comparing the previous identification to $(R_0^\square[[q-1]])^{(1)}/p$ shows a single Frobenius pullback lands the coaction of a Laurent crystal in the Laurentization of $((R_0^\square[[\tilde{p}]])^{(1)} \otimes_{R_0^\square[[\tilde{p}]]} R_0^\square[[q-1]])/p$. But then taking μ_{p-1} invariants further lands the coaction in $(R_0^\square[[\tilde{p}]])^{(1)}/p[1/\tilde{p}]$ as desired, giving the crystal. $\square$

4. FINITELY GENERATED UNIT LAURENT CRYSTALS

4.1. Basic constructions. Following [EK04b] and the more general version in [BL19] (which explains how to remove finiteness hypotheses), we first generalize the notion of finitely generated unit modules.

Definition 4.1. Let (A^+, I) be an orientable prism with $I = (d)$. Equip $A = A^+[1/d]_p^\wedge$ with the analytic ring structure in Definition A.11, and we use $D_\bullet(A)$ as defined there. Note A has a natural Frobenius φ induced by the Frobenius on A^+ (since $\varphi(d) \equiv d^p \pmod{p}$).

Write $A\langle F \rangle$ for

$$\widehat{\bigoplus_{i \geq 0} AF^i},$$

where $Fx = \varphi(x)F$. We put $D_\bullet(A\langle F \rangle)$ for solid modules over this noncommutative analytic ring, with the analytic ring structure induced from A .

For $M \in D_\bullet(A\langle F \rangle)$, we say M is *unit* if

$$\varphi_A^* M := A \otimes_{A, \varphi_A}^{L, \bullet} M \longrightarrow M$$

given by $a \otimes m \mapsto aF_M(m)$ is an equivalence. A unit object is *finitely generated unit* if it is perfect over $A\langle F \rangle$; these objects form the full subcategory $D_{\text{fgu}}(A)$. Here, we mean perfect in the sense that it lies in the thick subcategory generated by $A\langle F \rangle$ as we work over a noncommutative base.

We remark that we can naturally view these as certain unit Frobenius modules, so $D_{\text{fgu}}(A) \subset D_\bullet(A)^{\varphi=1}$ as a full subcategory. In fact one can show the category on a fixed prism is insensitive to the solid structure, but we will still remember it so that our crystals live in an ambient category which is a stack.

Lemma 4.2. *Let $(A^+, (d))$ be an orientable prism, and let $A = A^+[1/d]_p^\wedge$ with the analytic ring structure from Definition A.11. Then*

$$\text{Thick}_{A\langle F \rangle} D(A\langle F \rangle) \simeq \text{Thick}_{A\langle F \rangle} D_\bullet(A\langle F \rangle)$$

with inverse given by $M \mapsto M()$.*

The solid modules in the target thick subcategory are all p -completely nuclear.

Proof. The comparison is exact and carries $A\langle F \rangle$ to the corresponding solid module. To show the equivalence it therefore suffices to compute $\text{RHom}_{D_\bullet(A\langle F \rangle)}(A\langle F \rangle, A\langle F \rangle)$ and similarly on the module side and check that they are identified.

We can compute

$$\text{RHom}_{D_\bullet(A\langle F \rangle)}(A\langle F \rangle, A\langle F \rangle) \simeq \text{RHom}_{D_\bullet(A)}(A, A\langle F \rangle) \simeq A\langle F \rangle.$$

To justify this, by derived p -completeness it suffices to check equivalences modulo p^n , which makes the argument slightly easier as then $A\langle F \rangle // p^n = (A // p^n)[F]$. The first equivalence uses the free/forgetful adjunction, and the second equivalence is then clear. This then matches the corresponding calculation in ordinary modules, so it follows $\text{Thick}_{A\langle F \rangle} D(A\langle F \rangle) \simeq \text{Thick}_{A\langle F \rangle} D_\bullet(A\langle F \rangle)$.

Now we address the nuclearity claim. The unit A^+ of $D_\bullet(\text{Spa}(A^+, A^+))$ is nuclear. Since A is the filtered colimit of multiplication by d in derived p -complete solid modules, it is p -completely

nuclear. The p -completely nuclear objects are closed under colimits, so $A\langle F \rangle = \widehat{\bigoplus_{i \geq 0} AF^i}$ is p -completely nuclear. Nuclearity is preserved under the operators cutting out a thick subcategory, for example retracts or finite limits. $\square$

Definition 4.3. Let $(X, \mathcal{M})$ be a bounded fs log p -adic formal scheme. A crystal

$$\mathcal{E} \in D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$$

is finitely generated unit if for every object $(A, I, M_{\text{Spf } A})^a$ of $(X, \mathcal{M})_{\Delta}$, its evaluation belongs to $D_{\text{fgu}}(A[1/I]_p^{\wedge})$ in the sense of Definition 4.1.

We write $D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ for the full subcategory of finitely generated unit crystals, and $\text{Mod}_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ for its intersection with the heart of $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$.

Remark 4.4. Let B be a p -adically complete associative ring (in which p is central), and let M be a derived p -complete left B -module. Then

$$M \in \text{Perf}(B) \iff (B/p) \otimes_B^L M \in \text{Perf}(B/p).$$

This can be seen by using [Sta26, Lemma 15.99.4, Tag 09AW] for $\{(B/p^n) \otimes_B^L M\}_{n \geq 1}$. Passing through Lemma 4.2 we can run the same argument in the solid setting for $A\langle F \rangle$, thus detecting the finitely generated unit condition modulo p .

When $(X, \mathcal{M})$ satisfies Assumption 3.1 over $W(k)[\zeta_p]$, Remark 4.15 shows that it suffices to test Definition 4.3 on the log q -de Rham prisms associated to framed log affines.

Our first main goal will be to give an explicit local description of this category, as well as deduce strict étale descent. We note that while $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$ is a strict étale stack (by Corollary A.18), it is not immediate that the finitely generated unit subcategory is also a strict étale stack. While ordinary dualizable objects can be detected termwise in a limit, because we work over $A\langle F \rangle$ we use the thick subcategory generated by the unit (which satisfies no such property). Instead, descent of the fgu condition will use its characterization by coherent roots, which we now study.

Definition 4.5. Let A be as in 4.1, and assume that A is Noetherian regular. If M is a unit Frobenius module over A , a *root* of M is a coherent A -submodule $M_0 \subset M$ together with an injective map $\beta : M_0 \rightarrow \varphi_A^* M_0$ whose iterates identify

$$M \simeq \text{colim}_n \varphi_A^{*n} M_0.$$

It is *minimal* if it has no proper subroot. A distinguished minimal root is a minimal root contained in every other root of M as a submodule of the same ambient unit module.

One can make the analogous definitions in $\text{QCoh}_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$. When referring to a root of a module we mean the former, and when referring to a root of a crystal we mean the latter.

The essential idea of the proof of strict étale descent is to characterize the finitely generated unit condition in terms of distinguished minimal roots *as crystals* in Proposition 4.10, and use that coherent crystals satisfy strict étale descent to glue these. The uniqueness forces these roots to agree on overlaps.

Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine in the sense of Definition 3.2 and put $A = R_0^\square((q-1))_p^\wedge$, the Laurent version of its logarithmic q -de Rham prism. This ring is Noetherian regular, and in particular we can extract the following more practical criterion for verifying the finitely generated unit condition.

Lemma 4.6. *For $A = R_0^\square((q-1))_p^\wedge$, a unit $A\langle F \rangle$ -complex M is perfect over $A\langle F \rangle$ if and only if M is t -bounded and each $H^i(M)$ is finitely generated as a left $A\langle F \rangle$ -module.*

Proof. Supposing M is perfect over $A\langle F \rangle$, then so is $M//p$. Then [BL19, Proposition 11.3.9] shows all $H^i(M//p)$ are finitely generated over $\bar{A}[F]$. Now

$$H^i(M)/p \hookrightarrow H^i(M//p)$$

is a unit submodule, hence finitely generated. Then lifting generators to get a map $A\langle F \rangle^{\oplus n} \rightarrow H^i(M)$ and applying derived Nakayama to the derived p -complete cokernel for surjectivity shows $H^i(M)$ is finitely generated.

Conversely, suppose all $H^i(M)$ are finitely generated over $A\langle F \rangle$. Then $H^i(M)/p$ is as well, so to conclude we only need to show $H^{i+1}(M)[p]$ is as well, using the exact sequences

$$0 \longrightarrow H^i(M)/p \longrightarrow H^i(M//p) \longrightarrow H^{i+1}(M)[p] \longrightarrow 0.$$

Indeed [BL19, Proposition 11.3.9] then shows $M//p$ is finitely generated unit, and we already know from Remark 4.4 that it can be detected modulo p .

Thus we reduce to proving the assertion that if M is a finitely generated left module over $A\langle F \rangle$ that $M[p]$ is too. For a finitely generated unit module, $M[p^\infty] = M[p^\ell]$ for some ℓ , as otherwise upon considering

$$\mathrm{im}(M[p^\ell] \rightarrow (M/pM))$$

we would get an infinite ascending chain of finitely generated unit submodules (which isn't possible by root considerations and Noetherianity of $\bar{A}$). Once the chain stabilizes derived p -completeness forces $M[p^\infty] = M[p^\ell]$. We then consider the finite filtration $M[p] \cap p^i M$ for $i \leq \ell$. The graded pieces are unit subquotients of M/pM , hence finitely generated unit. Since the filtration is finite the desired claim follows. $\square$

In Theorem 4.7, we will identify finitely generated unit crystals with certain (φ, Γ) -modules.

A useful aspect is that we may ignore solid modules locally and work with continuous semilinear Γ -actions on topological modules. By $D_{\mathrm{fgu}}(A)^{h\Gamma}$, we mean unit objects in

$$\lim_{[s] \in \Delta} \mathrm{Thick}_{C(\Gamma^s, A\langle F \rangle)} D(C(\Gamma^s, A\langle F \rangle))$$

with no solid structures involved. Here $A\langle F \rangle$ has its natural topology (induced from A and the completed direct sum). Note $C(\Gamma^s, A) \widehat{\otimes}_A A\langle F \rangle \simeq C(\Gamma^s, A\langle F \rangle)$ as topological rings.

With this, we can prove the following explicit description of $D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge)$.

Theorem 4.7. *Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine in the sense of Definition 3.2 and put $A = R_0^\square((q-1))_p^\wedge$. Evaluation on the log q -de Rham prism $(R_0^\square[[q-1]], [p]_q, \mathcal{M})^a$ induces an equivalence*

$$\rho_A^{\Gamma,*} : D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \xrightarrow{\sim} D_{\mathrm{fgu}}(A)^{h\Gamma}.$$

Proof. First, we note that the functor is fully faithful. This follows from Proposition 3.13, but for the solid version of the target given by

$$D_\square(A)^{h\Gamma, \varphi=1} = \lim_{[s] \in \Delta} D_\square(C(\Gamma^s, A))^{\varphi=1}.$$

The essential image, by definition of $D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$, satisfies the property that evaluation on the log prism $C(\Gamma^s, A^+)$ we get a unit Frobenius module in $\mathrm{Thick}_{C(\Gamma^s, A\langle F \rangle)} D(C(\Gamma^s, A\langle F \rangle))$. By Lemma 4.2, it follows that equivalently the functor lands in $D_{\mathrm{fgu}}(A)^{h\Gamma}$, by which we mean unit objects in

$$\lim_{[s] \in \Delta} \mathrm{Thick}_{C(\Gamma^s, A\langle F \rangle)} D(C(\Gamma^s, A\langle F \rangle)).$$

Thus we obtain the desired fully faithful functor

$$\rho_A^{\Gamma,*} : D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_{\mathrm{fgu}}(A)^{h\Gamma}.$$

Now we turn to essential surjectivity. We first prove essential surjectivity onto the heart of the target modulo p . Put $\bar{A} = A/p$, and let M be a finitely generated unit $\bar{A}$ -module with continuous semilinear Γ -action.

Since $\bar{A}$ is regular and F -finite, [Bli08, Corollary 2.26] gives a distinguished minimal root

$$\beta : M_{\min} \hookrightarrow \varphi^* M_{\min}.$$

Every Γ -translate is again a root. Since it is a distinguished minimal root, it becomes Γ -equivariant and Theorem 3.5 lifts it to a coherent Laurent crystal $\mathcal{E}_0$. Its unitalization $\mathcal{E}$ is necessarily finitely generated unit⁶, and satisfies $\rho_A^* \mathcal{E} \simeq M$, giving the desired preimage for objects in the heart of the target $D_{\mathrm{fgu}}(A)^{h\Gamma}$ modulo p .

This suffices to prove essential surjectivity. Since evaluation on the log prism A^+ is exact, in particular compatible with shifts and fibers, a Postnikov dévissage (see [Hau24, Lemma 5.4.3]) on the target upgrades essential surjectivity on the heart modulo p to essential surjectivity on the derived categories. Thus evaluation is an equivalence modulo p . Lemma 2.3 now gives the claim. $\square$

Remark 4.8. On the heart one may again work classically, instead asking that the Γ action is continuous for the weak topology when viewed as a finitely generated unit $A\langle F \rangle$ -module. One may use the same argument as Lemma 3.15 (one must use that M is finitely generated unit to deduce K has the subspace topology in the proof, but otherwise it is the same). In later arguments we will only need the easy direction, namely that from the cobar construction one obtains an action

⁶If $B = B^+[1/I]_p^\wedge$ for a log prism $(B^+, I, M_{\mathrm{Spf} B^+})^a$, let M_0, M be the evaluations of $\mathcal{E}_0, \mathcal{E}$. We have

$$M \simeq [B\langle F \rangle \otimes_B^{L, \square} M_0 \xrightarrow{1-\tilde{\beta}} B\langle F \rangle \otimes_B^{L, \square} M_0]$$

where $\tilde{\beta}$ is induced by $\beta : M_0 \rightarrow \varphi^* M_0$, showing $\mathcal{E}$ is finitely generated unit.

continuous for the weak topology or $M \rightarrow C(\Gamma, A\langle F \rangle) \otimes_{A\langle F \rangle, p_0} M \rightarrow C(\Gamma, M)$ is continuous. The first map as explained in Lemma 3.15 is automatically continuous, and for the second to deduce just continuity we do not need K to have the subspace topology (or to even use it at all).

Showing t -exactness is more subtle, because when working in the solid setting it is unclear if $\rho_A^{\Gamma,*}$ is t -exact when we use the larger source category $D_{\blacksquare}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. However the finitely generated unit condition is sufficient to deal with this issue.

Lemma 4.9. *In Theorem 4.7, the t -structure on solid Laurent crystals restricts to the source $D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ and $\rho_A^{\Gamma,*}$ is t -exact.*

Proof. We note that in this proof of essential surjectivity in Theorem 4.7 implicitly the preimages constructed are all in the heart of $D_{\blacksquare}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. Indeed mod p the preimage is a unitalization of a coherent root, and coherent crystals are in the heart of the solid t -structure by Corollary A.19. The analogous procedure, instead run for exact sequences in the heart of $D_{\mathrm{fgu}}(A)^{h\Gamma}$ to iteratively write heart objects as extensions of p^{ℓ} -torsion objects (and taking a limit), constructs preimages in the heart by full faithfulness.

The desired claim is completely formal using the inverse equivalence in Theorem 4.7. Indeed, we have seen the inverse gives a t -exact functor

$$D_{\mathrm{fgu}}(A)^{h\Gamma} \longrightarrow D_{\blacksquare}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$$

since the preimages constructed lie in the heart. But then for any finitely generated unit Laurent crystal $\mathcal{E}$ in the essential image of this functor this implies its cohomology sheaves $H^i(\mathcal{E})$ are also finitely generated unit, so the t -structure restricts to the essential image of $D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. This argument also shows $\rho_A^{\Gamma,*}$ is t -exact. $\square$

By applying Theorem 3.5 we can prove a root criterion for the finitely generated unit condition.

Proposition 4.10. *Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine. A crystal*

$$\mathcal{E} \in D_{\blacksquare}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$$

is finitely generated unit if and only if it is t -bounded and each crystal $H^i(\mathcal{E} // p)$ admits a distinguished minimal root in $\mathrm{Coh}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}$.

Proof. Combining Theorem 4.7, Lemma 4.6 and Remark 4.4 to work modulo p , we see that each $H^i(\mathcal{E} // p)$ admitting a distinguished minimal root in $\mathrm{Coh}((\mathrm{Spf} R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}$ is sufficient. Indeed we see the unitalization of a coherent crystal root is a finitely generated unit Laurent crystal, by the same argument as in the footnote 6.

Conversely, take a finitely generated unit Laurent crystal. Reducing modulo p using Theorem 4.7 and Lemma 4.6 we reduce to the case where $\mathcal{E}$ corresponds to $M \in \mathrm{Mod}_{\mathrm{fgu}}(A)_{p=0}^{h\Gamma}$. Here we also implicitly use Lemma 4.9. By the previous argument in Theorem 4.7, by [Bli08, Corollary 2.26] there is a distinguished Γ -equivariant minimal root M_0 of M . Using Theorem 3.5 this root arises from a crystal $\mathcal{E}_0$. Together with Proposition 3.13, this theorem identifies roots as Laurent crystals with Γ -equivariant roots of M . Since M_0 has the stronger property of being a distinguished minimal root, the claim follows. $\square$

4.2. Strict étale descent. In Proposition 4.10, we have already produced the crucial observation we will need for descent: the strategy will be to show that for a strict étale cover, if we know the Laurent crystals glue then so do distinguished minimal roots, as uniqueness forces them to agree on overlaps. This glosses over an important point though, which is that we need pullback stability for minimal roots (by virtue of everything being regular F-finite, [Bli08] will again make these minimal roots automatically distinguished minimal).

Lemma 4.11. *Let A be a regular local F-finite $\mathbf{F}_p$ -algebra with completion $\widehat{A}$, and let $N_0 \subset N$ be a root of a finitely generated unit $A[F]$ -module. Then N_0 is minimal if and only if $\widehat{A} \otimes_A N_0 \subset \widehat{A} \otimes_A N$ is minimal.*

Proof. The forward implication is [Bli04, Corollary 2.11]. Conversely, assume the completed root is minimal and let $N_1 \subset N_0$ be a subroot. Its completion is a subroot, hence $\widehat{A} \otimes_A N_1 = \widehat{A} \otimes_A N_0$.

Since $A \rightarrow \widehat{A}$ is faithfully flat, it follows that $N_1 = N_0$. Thus N_0 has no proper subroot, hence is minimal. $\square$

Lemma 4.12. *Let $\mathrm{Spa} B \rightarrow \mathrm{Spa} A$ be an étale morphism of regular affinoids over $k((q-1))$. Then pullback along this map sends minimal roots of finitely generated unit modules to minimal roots.*

Proof. We first note that A and B are both regular F-finite, so all of the hypotheses of [Bli08] are satisfied. For testing minimality, the results in [Bli08] give us several useful tools. Let us suppose that M is a finitely generated unit module over say A , with root $\beta : M_0 \rightarrow \varphi^* M_0$.

Following [Bli08, §2.4], Blickle constructs $\sigma_\beta^{-1}(M_0) \subset M_0$ which can be characterized as the smallest submodule M'_0 such that $\beta(M_0) \subset \varphi_A^* M'_0$. By [Bli08, Lemma 2.16] this construction commutes with localization, and a root is minimal in our sense precisely when $\sigma_\beta^{-1}(M_0) = M_0$.

This construction is given by an explicit formula, which we now explain. We let $D_A^{(1)} = \mathrm{End}_A(A^{(1)})$ be the algebra of right A -linear endomorphisms of the Frobenius twist $A^{(1)}$. Blickle gives a Frobenius descent functor

$$\sigma^{-1} : \mathrm{Mod}_{D_A^{(1)}} \longrightarrow \mathrm{Mod}_A$$

as the inverse to $A^{(1)} \otimes_A -$ (which is an equivalence since $A^{(1)}$ is finite locally free over A). We may compute $\sigma_\beta^{-1}(M_0)$ as $\sigma^{-1}(D_A^{(1)} \cdot \beta(M_0)) \subseteq \sigma^{-1} \sigma^* M_0 = M_0$.

Now we prove the claim. By localization compatibility, a root M_0 over A is minimal if and only if its localization at every maximal ideal is again minimal. We use here that $\sigma_\beta^{-1}(M_0) = M_0$ is an open condition, see [Bli08, Proposition 2.17]. The same then also holds for B . Together with Lemma 4.11, this shows that minimality may be checked on the *completed* local rings at the classical points of the rigid spaces.

Fix a maximal ideal $\mathfrak{m} \subset B$, let $x \in \mathrm{Spa}(B)$ be the corresponding rigid point, and put $\mathfrak{n} = \mathfrak{m} \cap A$. The morphism $\mathrm{Spa} B \rightarrow \mathrm{Spa} A$ factors near x into affinoid open immersions and a finite étale morphism. Since affinoid open immersions induce isomorphisms on completed local rings by [BGR84, 7.3.2/3], the map

$$\widehat{A}_{\mathfrak{n}} \longrightarrow \widehat{B}_{\mathfrak{m}}$$

is finite étale as an ordinary map of rings. Both rings are again regular and F -finite. It therefore remains to show that finite étale pullbacks preserve minimal roots.

Now we reduce to this situation, and can just assume the ring map $A \rightarrow B$ is finite étale. We would be done if we just showed the identity

$$\sigma_{\beta_B}^{-1}(B \otimes_A M_0) = B \otimes_A \sigma_{\beta}^{-1}(M_0),$$

as using minimality the right side simplifies to $B \otimes_A M_0$, meaning $B \otimes_A M_0$ is minimal. Using the explicit formula,

$$\sigma_{\beta_B}^{-1}(B \otimes_A M_0) = \sigma^{-1}(D_B^{(1)} \beta(B \otimes_A M_0)).$$

Now substitute $B \otimes_A D_A^{(1)} \simeq D_B^{(1)}$. This follows here since the relative Frobenius is an isomorphism, so $B \otimes_A A^{(1)} \simeq B^{(1)}$, and taking End gives the claim.

Applying this substitution yields

$$\sigma_{\beta_B}^{-1}(B \otimes_A M_0) = \sigma^{-1}(B \otimes_A D_A^{(1)} \cdot \beta(B \otimes_A M_0)) = \sigma^{-1}(D_A^{(1)} \cdot \beta(M_0)) \otimes_A B.$$

In the last step we have used that σ^{-1} commutes with étale base change, which again can be seen from $B \otimes_A D_A^{(1)} \simeq D_B^{(1)}$. Recalling the definition of $\sigma_{\beta}^{-1}(M_0)$ we have proved $\sigma_{\beta_B}^{-1}(B \otimes_A M_0) = B \otimes_A \sigma_{\beta}^{-1}(M_0)$ as desired. $\square$

Corollary 4.13. *Let $f : (\text{Spf } S, \mathcal{M}_{\text{Spf } S}) \rightarrow (\text{Spf } R, \mathcal{M}_{\text{Spf } R})$ be a strict étale map where the target is a framed log affine. If $\mathcal{E} \in D_{\text{fgu}}((\text{Spf } R, \mathcal{M}_{\text{Spf } R})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}$ with distinguished minimal root $\mathcal{E}_0$, then $f^*\mathcal{E}$ is finitely generated unit and $f^*\mathcal{E}_0$ is its distinguished minimal root.*

Proof. If $(\text{Spf } R, \mathcal{M}_{\text{Spf } R})$ has framing $\square$, give $(\text{Spf } S, \mathcal{M}_{\text{Spf } S})$ the framing $\square \circ f$. The claim follows by combining Lemma 4.12 and Proposition 4.10, using the description in Theorem 4.7 to apply the first lemma. $\square$

Now the proof of strict étale descent for $D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ follows.

Theorem 4.14. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $W(k)[\zeta_p]$. Then the assignment*

$$(X, \mathcal{M}) \mapsto D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$$

is a stack for the strict étale topology on such $(X, \mathcal{M})$.

Proof. By Corollary A.18, $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$ satisfies strict étale descent. The pullback is t -exact, and the cover may be taken finite, so boundedness is local. Let $\{\rho_i : U_i \rightarrow (X, \mathcal{M})\}_{i \in I}$ be a strict étale cover such that every $\rho_i^*\mathcal{E}$ is finitely generated unit.

Using Remark 4.4 and Proposition 4.10 we may reduce the claim to showing

$$\{\rho_i^*\mathcal{E} \in \text{Mod}_{\text{fgu}}((U_i)_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}\} \implies \{\mathcal{E} \in \text{Mod}_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}\}$$

for $\mathcal{E} \in \text{QCoh}_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$ killed by p , or that the condition can be detected locally on a strict étale cover. Note that the converse is obvious, because any test log prism for the U_i is also one for $(X, \mathcal{M})$, so the finitely generated unit condition on the right side already includes the conditions imposed on the left hand side.

By Lemma 3.4, after a finite strict étale refinement we may assume without loss of generality that every $U_i = \mathrm{Spf} R_i$ is a framed log affine. Put $U_{ij} = U_i \times_{(X, \mathcal{M})} U_j$ with projections p_i, p_j ; each affine open of U_{ij} inherits a framing through either projection. Proposition 4.10 gives distinguished minimal roots $\mathcal{R}_i \subset \rho_i^* \mathcal{E}$. Corollary 4.13, applied with each framing, makes $p_i^* \mathcal{R}_i$ and $p_j^* \mathcal{R}_j$ distinguished minimal roots on any affine open of U_{ij} . We may further cover U_{ij} by framed log affines.

Their uniqueness in Proposition 4.10, or what we called ‘distinguished’, gives

$$p_i^* \mathcal{R}_i = p_j^* \mathcal{R}_j \subset \mathcal{E}|_{U_{ij}}.$$

Indeed by the remark after Corollary A.19 one has descent for coherent Laurent crystals $\mathrm{Coh}^\varphi((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ with a lax Frobenius. Thus we can check agreement locally on the chosen open cover of U_{ij} by framed log affines, which also shows agreement of the root maps. These equalities also satisfy the cocycle condition on triple overlaps, which is all we need to check as we work with coherent Laurent crystals.

Strict étale descent for $\mathrm{Coh}^\varphi((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)_{p=0}$ and $\mathrm{D}_\bullet^\varphi((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)_{p=0}^\heartsuit$ produces a global coherent root $\mathcal{E}_0$. Restricting any subroot or any other root to the cover shows that this root is again distinguished minimal. It is clear the unitalization remains $\mathcal{E}$ (as the claim is local and pullbacks commute with colimits). By the argument in Footnote 6 we deduce $\mathcal{E}$ is finitely generated unit from the existence of a root. $\square$

Remark 4.15. Combined with Theorem 4.7, this shows that the finitely generated unit condition can equivalently be tested upon evaluation by a covering family of log q -de Rham prisms as claimed in the introduction.

Let us record an important fact we showed in the proof of the theorem.

Corollary 4.16. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $W(k)[\zeta_p]$. A crystal*

$$\mathcal{E} \in \mathrm{D}_\bullet((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\varphi=1}$$

is finitely generated unit if and only if it is t -bounded and, for every ℓ , the unit Laurent crystal $H^\ell(\mathcal{E} // p)$ admits a unique global minimal coherent root.

Proof. The forward implication is the construction in the proof of Theorem 4.14, applied to the cohomology sheaves of $\mathcal{E} // p$ using Proposition 4.10 locally. The converse follows from the first paragraph of the proof of Proposition 4.10 which argues on a framed log affine why it suffices to know $H^\ell(\mathcal{E} // p)$ admit roots. Then as we know the property of being finitely generated unit is strict étale local by Theorem 4.14 the general claim follows. $\square$

Remark 4.17. This is an analogue of the root criterion for finitely generated unit modules on a k -scheme X . By [EK04b, Theorem 6.1.3], a finitely generated unit $\mathcal{O}_{F, X}$ -module admits a coherent root on every quasicompact open; as X is quasicompact, the root is global.

Such a root need not carry a crystal structure like in the previous corollary, but it turns out there always exists a root carrying such a structure. This seems to not be in the literature but follows

from a short argument. Cartier descent (see [Kat70, Theorem 5.1]) gives a fully faithful functor

$$\mathrm{QCoh}(X^{(1)}) \longrightarrow \mathrm{QCoh}((X/k)_{\mathrm{cris}})$$

with essential image the crystals with vanishing p -curvature. If $\mathrm{Fr} : X \rightarrow X^{(1)}$ is relative Frobenius, it sends $\mathcal{F}$ to $(\mathrm{Fr}^*\mathcal{F}, \nabla_{\mathrm{can}})$, where $(\mathrm{Fr}^*\mathcal{F})^{\nabla_{\mathrm{can}}=0} \simeq \mathcal{F}$.

Write the absolute Frobenius as $F_X = \pi \circ \mathrm{Fr}$, with $\pi : X^{(1)} \rightarrow X$. Given a coherent root $\beta : M_0 \rightarrow F_X^* M_0$, replace it by

$$F_X^* \beta : F_X^* M_0 \longrightarrow F_X^{*2} M_0.$$

Since F_X is finite flat, this map is injective and has the same unitalization. Moreover, $F_X^* M_0 = \mathrm{Fr}^* \pi^* M_0$ and $F_X^* \beta = \mathrm{Fr}^* (\pi^* \beta)$, so the map is horizontal for the canonical Cartier connections. It is therefore a coherent crystalline root. Uniqueness in Cartier descent identifies its unitalization with the crystal structure on M (similarly) induced by being an $\mathcal{O}_X$ -module with unit Frobenius.

Corollary 4.18. *Let $f : (Y, \mathcal{N}) \rightarrow (X, \mathcal{M})$ be a morphism between pairs satisfying Assumption 3.1 over $W(k)[\zeta_p]$. Then Lf^* preserves finitely generated unit crystals.*

Proof. Let $\mathcal{E}$ be a finitely generated unit crystal on $(X, \mathcal{M})_{\Delta}$, and let $(B, I, M_{\mathrm{Spf} B})^a$ be a log prism over $(Y, \mathcal{N})$. Upon composing with f , we can give it the structure of a log prism over $(X, \mathcal{M})$, and pullback of crystals gives a Frobenius equivariant equivalence

$$(Lf^*\mathcal{E})(B, I, M_{\mathrm{Spf} B}) \simeq \mathcal{E}(B, I, M_{\mathrm{Spf} B}).$$

Thus every evaluation of $Lf^*\mathcal{E}$ is perfect over $B[1/I]_p^\wedge \langle F \rangle$ by Definition 4.3. $\square$

4.3. The Kummer-étale realization and constructibility. In this section we construct a functor

$$T_{\mathrm{ét}} : D_{\mathrm{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\mathrm{op}} \longrightarrow D((X, \mathcal{M})_{\eta, \mathrm{prokét}}, \mathbf{Z}_p),$$

and show that its essential image is constrained to certain p -Kummer constructible sheaves that we will define shortly.

To make the definition of $T_{\mathrm{ét}}$, it is convenient to work on a basis of log perfectoids for the pro-Kummer étale site.

Lemma 4.19. *Let $(X, \mathcal{M})$ be a locally Noetherian fs log p -adic formal scheme. Then the log affinoid perfectoid objects form a basis B of $(X, \mathcal{M})_{\eta, \mathrm{prokét}}$ stable under fiber products.*

Proof. This is [DLLZ23, Propositions 5.3.11 and 5.3.12]. $\square$

For an object $U = \varprojlim_i U_i$ in the basis from the previous lemma, we write $\widehat{U} = \mathrm{Spa}(S, S^+)$ for its associated perfectoid space, equipped with the natural saturated log structure as in [DLLZ23, Remark 5.3.5].

Definition 4.20. Let $(X, \mathcal{M})$ be a locally Noetherian fs log p -adic formal scheme, and let $\mathcal{E}$ be a finitely generated unit Laurent crystal. Let $U \in B$ as in Lemma 4.19, with associated log perfectoid space

$$\widehat{U} = (\mathrm{Spa}(S, S^+), \mathcal{M}_S).$$

We let $\mathbb{A}_{S^+}$ be the log prism with underlying prism $(W(S^{+,b}), \ker \theta)$ and log structure induced by the tilted log structure on S^b . We define

$$T_{\text{ét}}(\mathcal{E})(U) := \text{RHom}_{D_{\blacksquare}^{\varphi}(W(S^b))}(\mathcal{E}(\mathbb{A}_{S^+}), W(S^b)) \in \widehat{D}(\mathbf{Z}_p).$$

These assignments define a presheaf on B , which by “unfolding” (or right Kan extension) yields a presheaf on $(X, \mathcal{M})_{\eta, \text{prokét}}$ we denote by $T_{\text{ét}}(\mathcal{E})$.

Remark 4.21. Our convention so far has been to use the strict site, and unfortunately these perfect prisms do not live in the strict site. However this is not an issue because Remark A.10 allows us to use the saturated log prismatic site and obtain the same category of crystals.

The category $D_{\blacksquare}(\mathbb{A}_{\text{inf}}(S)) = D_{\blacksquare}(W(S^b)) := \lim_n D_{\blacksquare}(\text{Spa}(\mathbb{A}_{\text{inf}}(S)/p^n, \mathbb{A}_{\text{inf}}(S^+)/p^n))$, with $\text{Spa}(\mathbb{A}_{\text{inf}}(S)/p^n, \mathbb{A}_{\text{inf}}(S^+)/p^n)$ given its natural analytic ring structure from the Huber pair as in Definition A.11. One can see $\mathbb{A}_{\text{inf}}(S) = \mathbb{A}_{\text{inf}}(S^+)[1/\xi]_p^{\wedge}$ where $\ker \theta = (\xi)$ as both are strict p -rings.

Before proceeding further, we show that the output is actually a sheaf.

Lemma 4.22. *Let $(X, \mathcal{M})$ be a locally Noetherian fs log p -adic formal scheme. Definition 4.20 yields a well-defined functor*

$$T_{\text{ét}} : D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\text{op}} \longrightarrow D((X, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p).$$

Proof. We will actually check hyperdescent, which is made easier as hypersheaves on the basis B from Lemma 4.19 are the same as hypersheaves on $(X, \mathcal{M})_{\eta, \text{prokét}}$.

For $U \in B$ let $U_{\bullet} \rightarrow U$ be a hypercover by basis objects in B , with associated perfectoid spaces $\widehat{U} = \text{Spa}(S, S^+)$ and $\widehat{U}_m = \text{Spa}(S_m, S_m^+)$. On the associated perfectoid spaces one gets a v -hypercover

$$\widehat{U}_{\bullet} \longrightarrow \widehat{U}.$$

Here we use that the assignment $U \mapsto \widehat{U}$ commutes with fiber products, and the easier fact that pro-Kummer-étale covers induce v -covers.

Put $M = \mathcal{E}(\mathbb{A}_{S^+})$ and $M_m = \mathcal{E}(\mathbb{A}_{S_m^+})$. The desired descent claim is that

$$\lim_{[m] \in \Delta} \text{RHom}_{D_{\blacksquare}^{\varphi}(\mathbb{A}_{\text{inf}}(S_m))}(M_m, \mathbb{A}_{\text{inf}}(S_m)) \simeq \text{RHom}_{D_{\blacksquare}^{\varphi}(\mathbb{A}_{\text{inf}}(S))}(M, \mathbb{A}_{\text{inf}}(S)).$$

We can use the extension and restriction adjunction to make all RHom s in $D_{\blacksquare}^{\varphi}(\mathbb{A}_{\text{inf}}(S))$, simplifying the left side to

$$\lim_{[m] \in \Delta} \text{RHom}_{D_{\blacksquare}^{\varphi}(\mathbb{A}_{\text{inf}}(S))}(M, \mathbb{A}_{\text{inf}}(S_m)).$$

Thus the needed claim is actually that $\mathbb{A}_{\text{inf}}(S) \simeq \text{R} \lim_{[m] \in \Delta} \mathbb{A}_{\text{inf}}(S_m)$. The claim can be checked modulo p , where it reduces to the v -acyclicity of the structure sheaves on $\text{Spa}(S_m^b, S_m^{+,b})$ shown in [Sch26a, Theorem 1.2]. Here we must also keep track of Frobenius and solid structures, but this much is clear. $\square$

It is easy to see that for every morphism $f : (Y, \mathcal{N}) \rightarrow (X, \mathcal{M})$ between pairs satisfying Assumption 3.1, the construction is compatible with pullback:

$$T_{\text{ét}, Y}(L f^* \mathcal{E}) \simeq L f_\eta^* T_{\text{ét}, X}(\mathcal{E}).$$

As previously claimed, the functor $T_{\text{ét}}$ actually lands in the full subcategory of p -Kummer sheaves which we now define. Recall that for a log geometric point $\bar{x} \in (X, \mathcal{M})_\eta$ we have an exact sequence

$$0 \longrightarrow I_{\bar{x}}^{\log} \longrightarrow \pi_1^{\text{prokét}}(\bar{x}_{\log}) \longrightarrow \pi_1^{\text{proét}}(\bar{x}) \longrightarrow 0$$

and one can compute $I_{\bar{x}}^{\log} = \text{Hom}(\overline{\mathcal{M}}_{\bar{x}}^{\text{gp}}, \widehat{\mathbf{Z}}(1))$. The prime-to- p part $I_{\bar{x}}^{\log, (p)'}$ is given by replacing $\widehat{\mathbf{Z}}(1)$ with its prime-to- p part.

Definition 4.23. Let $(X, \mathcal{M})$ be as in Assumption 3.1. We call $\mathcal{F} \in D_{\text{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$ *p -Kummer* if prime-to- p logarithmic inertia $I_{\bar{x}}^{\log, (p)'}$ acts trivially on every cohomology stalk of $\mathcal{F} // p^n$ at every log geometric point, for every $n \geq 1$.

For a derived lisse sheaf, this agrees with [IKY26, Definition 2, and Proposition 5]. The remainder of this section will prove that $T_{\text{ét}}(\mathcal{E})$ is constructible for the stratification induced by D , and as a byproduct that the result lands in the subcategory of p -Kummer sheaves.

We first need the following descent result. When dealing with these results on p -Kummer sheaves we allow a bit more generality, namely we can also allow $K/\mathbf{Q}_p$ ramified.

Lemma 4.24. *On pairs satisfying Assumption 3.1 over $\mathcal{O}_K$, the full subcategories of p -Kummer sheaves in*

$$D_{\text{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$$

form a stack for the strict étale topology.

We will denote these by $D_{\text{cons}, D}^{(b), \text{pKum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$ later.

Proof. We first check the claim without the p -Kummer condition imposed. This can be checked fairly directly. Let $f : (U, \mathcal{N}) \rightarrow (X, \mathcal{M})$ be a strict étale cover on the log formal schemes, and let $U(\bullet)$ be its Čech nerve.

Passing to adic generic fibers, this is compatible with fiber products so we can simply take log adic generic fibers of $U(\bullet)$ to obtain the corresponding Čech nerve on the log adic generic fiber. On the larger category of pro-Kummer-étale sheaves, trivially we have

$$D^{(b)}((X, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p) \simeq \varprojlim_{[n] \in \Delta} D^{(b)}((U^{(n)}, \mathcal{N}^{(n)})_{\eta, \text{prokét}}, \mathbf{Z}_p).$$

We also have such descent for $D_{\text{lisse}}^{(b)}$. By strictness, if $X_\eta^{(a)}$ and $U_\eta^{(a)}$ denote the loci where $\text{rk}_{\mathbf{Z}} \overline{\mathcal{M}}_{\bar{x}}^{\text{gp}} = a$ (respectively for $\mathcal{N}$), then

$$U_\eta^{(a)} = U_\eta \times_{X_\eta} X_\eta^{(a)}.$$

The logarithmic strata in the Čech are therefore compatible, and we also have descent for lisse sheaves on each stratum (the strata are obtained by base change, so the corresponding Čech

nerve again consists of strict étale maps). As we are interested in constructible sheaves for this stratification the claim follows.

For the categories $D_{\text{cons}, D}^{(b), p\text{Kum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$ with the p -Kummer condition imposed, note that strictness identifies the logarithmic inertia groups so the claim follows. $\square$

This has a useful implication: by Lemmas 3.4 and 4.24, constructibility for the logarithmic stratification induced by D may be checked on a framed log affine (as well as the p -Kummer condition). Thus for the desired result that $T_{\text{ét}}$ lands in the full subcategory $D_{\text{cons}, D}^{(b), p\text{Kum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$ we can now immediately assume that $(X, \mathcal{M})$, rather than simply satisfying Assumption 3.1, is a framed log affine.

In this case there is a natural perfectoid cover we will use. Let $(X, \mathcal{M})$ be a framed log affine as in Definition 3.2. Then the log formal scheme $\mathbb{T}_r^n$ induces a natural perfectoid cover

$$(\text{Spf } W(k) \langle \zeta_{p^\infty} \rangle \langle T_1^{1/p^\infty}, \dots, T_r^{1/p^\infty}, T_{r+1}^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty} \rangle)_\eta \longrightarrow (\mathbb{T}_r^n)_\eta.$$

Pulling this back along $\square_\eta$, we get a perfectoid cover

$$(X, \mathcal{M})_{\infty, \eta} \longrightarrow (X, \mathcal{M})_\eta$$

which is a quasi-pro-Kummer-étale torsor for Γ . When the framed log affine is over $W(k)[\zeta_p]$ as we usually assume, $\Gamma = (1 + p\mathbf{Z}_p) \ltimes \mathbf{Z}_p(1)^n$ as usual. We change the cyclotomic factor $1 + p\mathbf{Z}_p$ depending on the base, for example it is $\mathbf{Z}_p^\times$ when working over $W(k)$.

Associated to this perfectoid space the perfection of the log q de Rham prism or $R_0^\square[[q - 1]]_{\text{perf}}$. If we started over $W(k)[\zeta_p]$ this again has prismatic ideal $[p]_q$, and $\text{Spf}(R_0^\square[[q - 1]]_{\text{perf}}/[p]_q)_\eta$ is $(X, \mathcal{M})_{\infty, \eta}$. This identification is also compatible with the induced Γ action on the perfected prism and the Γ action on $(X, \mathcal{M})_{\infty, \eta}$.

Using this perfectoid cover, we can give another useful criterion to check the p -Kummer condition.

Lemma 4.25. *Let $(X, \mathcal{M})$ be a framed log affine with $X = \text{Spf } R$ and let $\mathcal{F}$ be derived p -complete in $D^{(b)}((\text{Spf } R, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p)$. For $\Lambda \subseteq \{1, \dots, r\}$, put*

$$(X, \mathcal{M})_{\infty, \eta, \Lambda} = (X, \mathcal{M})_{\infty, \eta} \times_{(X, \mathcal{M})_\eta} (X, \mathcal{M})_{\eta, \Lambda},$$

and let $\epsilon_\infty : ((X, \mathcal{M})_{\infty, \eta}, \mathcal{M}_{\infty, \eta})_{\text{prokét}} \rightarrow ((X, \mathcal{M})_{\infty, \eta})_{\text{proét}}$ be the natural morphism of sites. The following are equivalent:

- (1) $\mathcal{F}$ is constructible for the stratification induced by D and is p -Kummer;
- (2) Its pullback to $(X, \mathcal{M})_{\infty, \eta}$ is $\epsilon_\infty^* \mathcal{L}_\infty$ for some $\mathcal{L}_\infty \in D^{(b)}((X, \mathcal{M})_{\infty, \eta})_{\text{proét}}, \mathbf{Z}_p)$ which is constructible for the finite stratification $(X, \mathcal{M})_{\infty, \eta} = \bigsqcup_\Lambda (X, \mathcal{M})_{\infty, \eta, \Lambda}$.

Proof. Constructibility is clear, because the pro-Kummer-étale Galois cover $(X, \mathcal{M})_{\infty, \eta} \rightarrow (X, \mathcal{M})_\eta$ has the Galois group Γ respect the strata $(X, \mathcal{M})_{\infty, \eta} = \bigsqcup_\Lambda (X, \mathcal{M})_{\infty, \eta, \Lambda}$. In particular we may test lissity on the strata induced by D after pullback to $(X, \mathcal{M})_{\infty, \eta, \Lambda}$, and then constructibility along D after pullback to $(X, \mathcal{M})_{\infty, \eta} = \bigsqcup_\Lambda (X, \mathcal{M})_{\infty, \eta, \Lambda}$ by testing lissity on each stratum.

For the p -Kummer condition, we check stalkwise. The point is that the logarithmic inertia on the perfectoid cover $(X, \mathcal{M})_{\infty, \eta}$ extracts precisely the prime-to- p part on the base $(X, \mathcal{M})_\eta$: for

$\bar{y} \in (X, \mathcal{M})_{\infty, \eta}$ a log geometric point above $\bar{x} \in (X, \mathcal{M})_{\eta}$,

$$I_{\infty, \bar{y}}^{\log} \simeq \text{Hom}(\mathbf{Z}[1/p]^{|\Lambda|}, \widehat{\mathbf{Z}}(1)) = I_{\bar{x}}^{\log, (p)'}$$

since $I_{\bar{x}}^{\log, (p)'}$ is the prime-to- p part of $\text{Hom}(\overline{\mathcal{M}}_{\bar{x}}^{\text{gp}}, \widehat{\mathbf{Z}}(1))$. The condition we ask, that the pullback is of the form $\epsilon_{\infty}^* \mathcal{L}_{\infty}$, is equivalent to saying this acts trivially. $\square$

We let $T_{\text{ét}}^{\text{perf}}$ denote the *schematic* solution functor introduced in Theorem 2.6. We let $(R_0^{\square})_{\infty, \eta}$ denote the associated perfectoid ring for $(X, \mathcal{M})_{\infty, \eta}$ over $\mathbf{Q}_p$, and in what follows we will apply $T_{\text{ét}}^{\text{perf}}$ to $\text{Spec}(R_0^{\square})_{\infty, \eta}^b$. For a finitely generated unit module M over $R_0^{\square}((q-1))_p^{\wedge}$, we use the convention

$$T_{\text{ét}}^{\text{perf}}(M) := T_{\text{ét}}^{\text{perf}}\left(M \otimes_{R_0^{\square}((q-1))_p^{\wedge}}^L W((R_0^{\square})_{\infty, \eta}^b)\right),$$

as it does not literally make sense as an input before base changing.

The basic structure of the argument for constructibility will be to produce a factorization of $T_{\text{ét}}$ in Lemma 4.30 on a framed log affine involving $T_{\text{ét}}^{\text{perf}}$. With the input of $T_{\text{ét}}$ regarded as an object in $D_{\text{fgu}}(R_0^{\square}((q-1))_p^{\wedge})^{h\Gamma}$ by Theorem 4.7, we can ultimately reduce to objects M in the heart killed by p and argue that they have Γ -equivariant minimal roots M_0 as in the proof of Theorem 4.7. Then looking at the root map

$$\beta : M_0 \hookrightarrow \varphi^* M_0$$

and the cokernel of β all are finite Γ -equivariant modules. We constrain these by studying their Fitting ideals which are necessarily stable under Γ , and classifying the possibilities enough to force the root map to be an equivalence on each stratum (by forcing $M_0, \varphi^* M_0$ and $\text{coker } \beta$ to be vector bundles through the usual Fitting criterion in [Sta26, Lemma 15.8.8, Tag 07ZD]). One then deduces using the factorization in Lemma 4.30 the desired constructibility, as using Katz's theorem the result on each stratum is lisse at the perfectoid level (whereupon the criterion in Lemma 4.25 is useful).

We begin with a classification of stable ideals.

Lemma 4.26. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine, and put $\bar{R} = R_0/p$. If $\mathfrak{a} \subset \bar{R}^{\square}((q-1))$ is an ideal stable under $\mathbf{Z}_p(1)^n \subset \Gamma$, then, on each connected component of*

$$\text{Spa } \bar{R}^{\square}((q-1)) \setminus V(T_1 \cdots T_r)$$

the image of $\mathfrak{a}[1/T_1, \dots, 1/T_r]$ is either zero or the unit ideal.

Moreover, the same conclusion holds for the scheme $\text{Spec } \bar{R}^{\square}((q-1))[1/T_1, \dots, 1/T_r]$.

Proof. Put $X = \text{Spa } \bar{R}^{\square}((q-1))$ and $X^{\circ} = X \setminus V(T_1 \cdots T_r)$. Due to the framing $\square$, as rigid analytic spaces over $k((q-1))$ we have an étale map

$$X \longrightarrow \text{Spa } k((q-1)) \langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1} \rangle.$$

The target is reduced, hence so is X . Through this étale map, we also get an action of $\mathbf{Z}_p(1)^n$ on X which preserves X° . On the base, elements $x \in \mathbf{Z}_p(1)_i$ of the i th copy of $\mathbf{Z}_p(1)$ act by $T_i \mapsto q^{p^x} T_i$ and fixes the other coordinates.

Put $\mathcal{I} = \mathfrak{a} \mathcal{O}_{X^{\circ}}$, and let x be a classical point for which $\mathcal{I}_x$ is a proper ideal. Since $\mathcal{O}_{X^{\circ}, x}$ is a local ring, this means $\mathcal{I}_x$ is contained in $\mathfrak{m}_x \mathcal{O}_{X^{\circ}, x}$ and so all $f \in \mathfrak{a}$ vanish at x . Up to replacing

$k((q-1))$ by the residue field $\kappa(x)/k((q-1))$, the rigid analytic inverse function theorem will identify some neighborhood of x with a polydisk. Whether or not $\mathcal{I}_x$ vanishes can be checked after this extension, so without loss of generality we can assume x is a rational point with a polydisk neighborhood.

Now we use $\mathbf{Z}_p(1)^n$ stability of $\mathfrak{a}$. Considering the elements $\gamma_\alpha = (p^{\alpha_1}, \dots, p^{\alpha_n}) \in \mathbf{Z}_p(1)^n$, stability means for $f \in \mathfrak{a}$ that $\gamma_\alpha \cdot f \in \mathfrak{a}$. They all vanish at x , so we will also get vanishing at $q^{p^{\alpha_i+1}}T_i(x)$. For $\min_i \alpha_i \gg 0$ these live in the polydisk around x , by continuity of the action of $\mathbf{Z}_p(1)^n$. Since we work in positive characteristic we get

$$q^{p^{\alpha+1}} - 1 = (q-1)^{p^{\alpha+1}} \longrightarrow 0,$$

so we have produced an accumulating grid of zeroes in a polydisk around x . Applying the single variable rigid analytic identity theorem repeatedly, as in [Pen25, Lemma 3.9], shows that a finite set of generators of $\mathfrak{a}$ vanishes on a neighborhood of x . Thus $\mathcal{I}_x = 0$ if $\mathcal{I}_x$ is a proper ideal. In particular either $\mathcal{I}_x = \mathcal{O}_{X^\circ, x}$ or it is zero, and we have seen the locus where it is zero is open (i.e. it propagates to a polydisk around x). Thus on connected components it is either zero or the unit ideal.

This proves the first assertion. To deduce the claim for the scheme, the same claims hold for $\mathfrak{a}_\mathfrak{m}$ for maximal ideals $\mathfrak{m}$ of $\mathrm{Spec} \overline{R}^\square((q-1))[1/T_1, \dots, 1/T_r]$. Indeed the claim that $\mathcal{I}_x$ is zero or the unit ideal is simply about the base change to the completed local ring, which is faithfully flat over $\overline{R}^\square((q-1))[1/T_1, \dots, T_r]_\mathfrak{m}$ in our situation. Thus by faithfully flat descent, we deduce $\mathfrak{a}_\mathfrak{m}$ is zero or the unit ideal for maximal ideals in $\mathrm{Spec} \overline{R}^\square((q-1))[1/T_1, \dots, 1/T_r]$.

This is enough to deduce the desired claim that $\mathfrak{a}$ is 0 or 1 on connected components for the scheme. To see this, note that connected components are integral, and so reducing to the affine case we can use the fact that for a domain D that

$$D = \bigcap_{\mathfrak{m} \in \mathrm{mSpec} D} D_\mathfrak{m}.$$

The ideal $\mathfrak{a}$ is zero or one in each $D_\mathfrak{m}$, hence on D . The claim then follows. $\square$

Lemma 4.27. *Let $\overline{A} = \overline{R}^\square((q-1))$ for a framed log affine, and let $J_\Lambda = (T_i)_{i \in \Lambda}$ for some $\Lambda \subset \{1, \dots, r\}$. If $M \in D_{\mathrm{fgu}}(\overline{A})^{h\Gamma}$, then*

$$(\overline{A}/J_\Lambda) \otimes_{\overline{A}}^L M$$

is in $D_{\mathrm{fgu}}(\overline{A}/J_\Lambda)^{h\Gamma}$.

Proof. There is an induced Γ action because the ideal J_Λ is stable under Γ . The finitely generated unit claim is clear because

$$(\overline{A}/J_\Lambda)[F] \otimes_{\overline{A}[F]}^L M \simeq (\overline{A}/J_\Lambda) \otimes_{\overline{A}}^L M,$$

showing the result is in $\mathrm{Thick}((\overline{A}/J_\Lambda)[F])$. It is unit because J_Λ is Frobenius stable (as is any ideal in positive characteristic). $\square$

In the following proposition recall that $(R_0^\square)_{\infty, \eta}$ is the ring of functions of the affinoid perfectoid space $(\mathrm{Spf} R, \mathcal{M})_{\infty, \eta}$.

Proposition 4.28. *Let $(\mathrm{Spf} R, \mathcal{M}_{\mathrm{Spf} R})$ be a framed log affine with framing $\square$, and let $\mathcal{E}$ be an object of $D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M}_{\mathrm{Spf} R})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. Then $T_{\mathrm{ét}}^{\mathrm{perf}}(\mathcal{E}(R_0^{\square}((q-1))_p^{\wedge}))$ is constructible for the stratification of $\mathrm{Spec}(R_0^{\square})_{\infty, \eta}^b$ cut out by the divisor $D_{\infty} = \bigcup_i V(T_i^b)$.*

Proof. Because constructibility can be detected after reduction modulo p , from the start we can assume $\mathcal{E} \in D_{\mathrm{fgu}}((\mathrm{Spf} R, \mathcal{M}_{\mathrm{Spf} R})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_{p=0}$. As usual we put $\bar{A} = \bar{R}_0^{\square}((q-1))$ for the mod p reduction of $R_0^{\square}((q-1))_p^{\wedge}$.

Let $M = \mathcal{E}(R_0^{\square}((q-1))_p^{\wedge})$, which we may regard as an object $D_{\mathrm{fgu}}(\bar{A})^{h\Gamma}$ using Theorem 4.7. We defined $T_{\mathrm{ét}}^{\mathrm{perf}}(M)$ to be

$$T_{\mathrm{ét}}^{\mathrm{perf}}\left(M \otimes_{R_0^{\square}((q-1))_p^{\wedge}}^L W((R_0^{\square})_{\infty, \eta}^b)\right).$$

The input is in $D_{\mathrm{fgu}}(W((R_0^{\square})_{\infty, \eta}^b))$. For testing the claim, we may reduce to the case where M is a module (note the base change is flat, by flatness of Frobenius). To simply show that the output of $T_{\mathrm{ét}}^{\mathrm{perf}}$ in this case is lisse on the open stratum, or complement of D_{∞} , it would suffice to show that $M[1/T_1, \dots, 1/T_r]$ is a vector bundle over $\bar{A}[1/T_1, \dots, 1/T_r]$ – then the contravariant version of Katz’s theorem from Remark 2.2 shows the output is lisse.

Let us dispense with this case first. We regard everything in what follows as a module over $\bar{A}[1/T_1, \dots, 1/T_r]$ (i.e. we replace M with its base change to this ring). As we work modulo p , we can pick a Γ -equivariant minimal root $\beta : M_0 \rightarrow \varphi^* M_0$ for M (just as in the proof of Theorem 4.7 using [Bli08]). Now Lemma 4.26 provides some serious constraints: each $\mathrm{Fitt}_i(M_0)$ is $\mathbb{Z}_p(1)^n$ -stable as are Fitting ideals of $\varphi^* M_0$ and $\mathrm{coker} \beta$. Pick the minimal i so that $\mathrm{Fitt}_i(M_0) \neq 0$. One may test the desired constructibility claim locally, so we may reduce to the connected case and apply Lemma 4.26 to deduce this is the unit ideal. Finally applying [Sta26, Lemma 15.8.8, Tag 07ZD] makes M_0 locally free of rank i . We note $\varphi^* M_0$ has the same rank and is necessarily a vector bundle.

By the same argument, we also find $\mathrm{coker} \beta$ is a vector bundle of some possibly different rank. But then we have the situation where we have an exact sequence

$$0 \longrightarrow M_0 \xrightarrow{\beta} \varphi^* M_0 \longrightarrow \mathrm{coker} \beta \longrightarrow 0$$

consisting entirely of vector bundles, but the first two have the same rank. Thus $\mathrm{coker} \beta$ has rank zero, and β is an isomorphism. Thus $M = M_0$, and the contravariant version of Katz’s theorem tells us that $T_{\mathrm{ét}}^{\mathrm{perf}}(M)$ is lisse away from D_{∞} .

One can run exactly the same argument for any other stratum in the stratification induced by D_{∞} to see the result is lisse on this stratum. What we must do is consider $\bar{R}_0^{\square}((q-1))/J_{\Lambda}$ as in Lemma 4.27 and use this lemma to deduce the derived base change of M is finitely generated unit, and then we can invert any surviving T_i if desired. Running the same reduction to the heart and repeating the argument the result follows. $\square$

Remark 4.29. For the trivial log structure there is only the open stratum, so the same argument shows we obtain a derived lisse complex. In particular without log structures, the theory degenerates to the ordinary theory of (φ, Γ) -modules.

Let $\nu : (X_{\infty, \eta}^b)_{\text{ét}} \rightarrow \text{Spec}((R_0^{\square})_{\infty, \eta}^b)_{\text{ét}}$ be the support map, as used in [Bha25, Notation 6.2.1], and let $\lambda : ((X, \mathcal{M})_{\infty, \eta}^b)_{\text{proét}} \rightarrow ((X, \mathcal{M})_{\infty, \eta}^b)_{\text{ét}}$ be the natural morphism of sites. For a constructible $\mathbf{Z}_p$ -complex $\mathcal{F}$, set

$$\text{an}^* \mathcal{F} := \text{R} \varprojlim_m \lambda^* \nu^* (\mathcal{F} // p^m) \in D((X_{\infty, \eta}^b)_{\text{proét}}, \mathbf{Z}_p).$$

For a Γ -equivariant constructible complex $\mathcal{F}$, set

$$\text{an}_{\Gamma}^* \mathcal{F} := (\epsilon_{\infty}^* ((\text{an}^* \mathcal{F})^{\#}))^{h\Gamma} \in D((\text{Spf } R, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p),$$

where $(-)^{\#}$ denotes transport across the tilting equivalence (to characteristic zero) and $h\Gamma$ denotes descent along $(X, \mathcal{M})_{\infty, \eta} \rightarrow (X, \mathcal{M})_{\eta}$.

Lemma 4.30. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine with framing $\square : (\text{Spf } R, \mathcal{M}) \rightarrow \mathbb{T}_r^n$. Evaluation on $\Delta_{(R_0^{\square})_{\infty}}$ gives a commutative diagram*

$$\begin{array}{ccc} D_{\text{fgu}}((\text{Spf } R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge})^{\text{op}} & \xrightarrow{T_{\text{ét}}} & D((\text{Spf } R, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p) \\ \mathcal{E} \mapsto \mathcal{E}(\Delta_{(R_0^{\square})_{\infty}}) \downarrow & & \uparrow \text{an}_{\Gamma}^* \\ (D_{\text{fgu}}(W((R_0^{\square})_{\infty, \eta}^b))^{h\Gamma})^{\text{op}} & \xrightarrow{T_{\text{ét}}^{\text{perf}}} & D_{\text{cons}}^{(b)}(\text{Spec}(R_0^{\square})_{\infty, \eta}^b, \mathbf{Z}_p)^{h\Gamma} \end{array}$$

Proof. We showed already that the formula for $T_{\text{ét}}$ outputs a sheaf in $D((\text{Spf } R, \mathcal{M})_{\eta, \text{prokét}}, \mathbf{Z}_p)$ in Lemma 4.22, we may compute its output using the Čech nerve of $(X, \mathcal{M})_{\infty, \eta}$ which is a pro-Kummer-étale Γ -torsor over $(\text{Spf } R, \mathcal{M})_{\eta}$. In particular, once we use this specific cover to compute $T_{\text{ét}}(\mathcal{E})$ we get a factorization

$$T_{\text{ét}} : D_{\text{fgu}}((\text{Spf } R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge})^{\text{op}} \longrightarrow D((X, \mathcal{M})_{\infty, \eta, \text{prokét}}, \mathbf{Z}_p)^{h\Gamma}.$$

Setting $M_{\infty} = \mathcal{E}(\Delta_{(R_0^{\square})_{\infty}})$, the goal is therefore a Γ -equivariant identification

$$T_{\text{ét}}(\mathcal{E})(U) \xrightarrow{\sim} \epsilon_{\infty}^* ((\text{an}^* T_{\text{ét}}^{\text{perf}}(M_{\infty}))^{\#})(U)$$

for log perfectoids $U = (\text{Spa}(S, S^+), \mathcal{M}_S) \rightarrow (X, \mathcal{M})_{\infty, \eta}$. The equivariance is automatic once we do this functorially in U , as the Čech nerve of $(X, \mathcal{M})_{\infty, \eta}$ over $(X, \mathcal{M})_{\eta}$ is a bar construction in Γ .

To construct the map is straightforward. Let $\mathcal{F} = T_{\text{ét}}^{\text{perf}}(M_{\infty})$, and write $\mathcal{F}_S$ for its pullback to $\text{Spec } S^b$. The formula for $T_{\text{ét}}(\mathcal{E})(U)$ identifies it directly with $\text{R}\Gamma(\text{Spec } S_{\text{proét}}^b, \mathcal{F}_S)$. The right hand side $\epsilon_{\infty}^* ((\text{an}^* T_{\text{ét}}^{\text{perf}}(M_{\infty}))^{\#})(U)$ identifies with $\text{R}\Gamma(\text{Spa}(S^b, S^{+,b})_{\text{proét}}, \text{an}^* \mathcal{F}_S)$. Thus analytification, by functoriality, induces a comparison morphism.

By derived Nakayama, derived p -completeness reduces us to working with mod p coefficients. The comparison map between $\text{R}\Gamma(\text{Spa}(S^b, S^{+,b})_{\text{proét}}, \text{an}^* \mathcal{F}_S)$ and $\text{R}\Gamma(\text{Spec } S_{\text{proét}}^b, \mathcal{F}_S)$ is then an equivalence by [Ans22, Proposition 5.1]⁷, applied to $(S^b, S^{+,b})$, together with the étale versus proétale comparisons in [BS15, Corollary 5.1.6] and [Sch26a, Proposition 8.5(i)]. $\square$

⁷We note this comparison can be reproved using arc_{ϖ} descent results (which appeared later) for torsion étale cohomology from [BM21, Theorem 5.4 and Corollary 6.17], where ϖ is the tilted pseudouniformizer.

There is an instructive warning explained in [Bha25, Warning 6.2.8] related to the previous proof, which is that ν^* is *not* fully faithful on constructible sheaves. However this is compatible with the cohomological comparison we used, because the issue with full faithfulness comes from the fact that distinct constructible sheaves can have the same analytification, and in particular there is a nonzero constructible $\mathcal{F}$ with $\nu^*\mathcal{F} = 0$ (as can be constructed from the warning). However this particular example also has no cohomology, so it is perfectly compatible with the previous argument.

Remark 4.31. The factorization in Lemma 4.30 shows that $T_{\text{ét}}$ from Definition 4.20 is symmetric monoidal: for $T_{\text{ét}}^{\text{perf}}$, we showed it is symmetric monoidal in Theorem 2.6, and the remaining functors are symmetric monoidal. Note that one must also check that finitely generated unit modules are closed under $\otimes$, which can be done modulo p by Remark 4.4 and then using the roots provided by Corollary 4.16 and that fact that $\otimes$ commutes with colimits.

The same factorization proved in Lemma 4.30 controls the essential image of $T_{\text{ét}}$.

Proposition 4.32. *Every object in the essential image of $T_{\text{ét}}$ on $D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\text{op}}$ is a p -Kummer sheaf in $D_{\text{cons}, D}^{(b)}((X, \mathcal{M})_{\eta}, \mathbf{Z}_p)$.*

Proof. The result is automatically p -Kummer by Lemma 4.30 and Lemma 4.25.

For constructibility, we again use Lemma 4.30. Inspecting the diagram, it suffices to know that the result in $D_{\text{cons}}^{(b)}(\text{Spec}(R_0^{\square}_{\infty, \eta}, \mathbf{Z}_p)^{h\Gamma})$ is constructible for the stratification induced by $V(T_i^b)$. Indeed, analytification and untilting respect these strata, as does the Γ -action as the strata on $(X, \mathcal{M})_{\infty, \eta}$ are stable under the Γ action, and descend to the strata $(X, \mathcal{M})_{\Delta, \eta}$ induced by D_{η} on $(X, \mathcal{M})_{\eta}$.

Proposition 4.28 shows exactly this result, so the desired claim follows. $\square$

5. THE MAIN EQUIVALENCE

In this section we will prove the main theorem, which is that

$$T_{\text{ét}} : D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\text{op}} \longrightarrow D_{\text{cons}, D}^{(b), \text{pKum}}((X, \mathcal{M})_{\eta}, \mathbf{Z}_p)$$

is an equivalence.

We will give a brief overview of the strategy. In the first subsection, we focus on the open stratum $X_{\eta} \setminus D_{\eta}$. We will always denote the canonical inclusion map by $j : X_{\eta} \setminus D_{\eta} \rightarrow X_{\eta}$.

We first construct a log prismatic crystal we call $\mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}(*D)$ which corresponds to $j_!\mathbf{Z}_p$, and argue that

$$\text{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}(*D))^{\varphi=1, \text{op}}$$

is sent by $T_{\text{ét}}$ to derived lisse sheaves on $X_{\eta} \setminus D_{\eta}$. More precisely, the essential image is these but embedded in $D_{\text{cons}, D}^{(b), \text{pKum}}((X, \mathcal{M})_{\eta}, \mathbf{Z}_p)$ via the fully faithful functor $j_!$. This appears as Theorem 5.4.

To prove this result is actually the most nontrivial part of the main theorem. Briefly, the approach is to first argue strict étale descent to work on a framed log affine with framing $\square$ and associated log q de Rham prism $(R_0^{\square}[[q-1]], [p]_q, \mathcal{M})^a$. Once we are in this situation, we can use Lemma 4.30

and an analogue of Theorem 4.7 to reduce full faithfulness to Proposition 5.9, equivalently full faithfulness of

$$\mathrm{Perf}(\mathrm{R}_0^\square((q-1))_p^\wedge[1/T_1, \dots, 1/T_r]_p^\wedge)^{\varphi=1, h\Gamma} \longrightarrow \mathrm{Perf}(\mathbb{A}_{\mathrm{inf}}((\mathrm{R}_0^\square)_\infty)[1/\xi, 1/[T_1^b], \dots, 1/[T_r^b]]_p^\wedge)^{\varphi=1, h\Gamma}.$$

We reduce this to a calculation in Γ -cohomology which is then carried out using decompletion techniques from [KL16]. We must also use a result on full faithfulness of analytification from [Bha25].

For essential surjectivity, the important input comes from [IKY26]. Dualizing their correspondence gives an equivalence

$$\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\varphi=1, \mathrm{op}} \simeq D_{\mathrm{lis}}^{(b), \mathrm{pKum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p).$$

The idea is then to use the main result of [Lüt93], rephrased in [DLLZ23] to assert that a local system on the open stratum admits an extension to a Kummer-étale one. After a finite prime-to- p Kummer cover to make this a p -Kummer local system we may then produce a crystal $\mathcal{E}$ through [IKY26], at which point we can tensor with $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ and descend to a preimage in $\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1, \mathrm{op}}$.

The remainder of the argument to deduce the main theorem then largely takes an expected form. We prove a variant of Kashiwara's lemma to handle closed strata, and deduce a hollow variant of the theorem on the open stratum (for technical reasons). We can then prove the theorem by matching thick generators. More precisely, we know by this point that we have a well-defined functor

$$T_{\mathrm{ét}} : D_{\mathrm{figu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\mathrm{op}} \longrightarrow D_{\mathrm{cons}, D}^{(b), \mathrm{pKum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$$

and it is easy to see that it is exact. The target category is thickly generated by extensions by zero of local systems on locally closed log strata, so for essential surjectivity one must provide corresponding crystals on the source. We show these corresponding objects are also thick generators of the source, and moreover for maps between generators $T_{\mathrm{ét}}$ is fully faithful. The claim then follows.

5.1. The open stratum. We begin by constructing $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$, as explained in the overview. For an $\mathcal{O}_X^\times$ -torsor, let $\mathcal{L}_{\mathcal{O}_X^\times}(-)$ denote passage to the associated line bundle over $\mathcal{O}_X$. Recall also that part of the definition of the log structure $\mathcal{M}$ is a map

$$\alpha : \mathcal{M} \longrightarrow (\mathcal{O}_X, \cdot),$$

which we use in the following construction. We remark that this is similar to [Tsu99], where such a crystal is constructed for the log crystalline site.

Definition 5.1. Let $\mathcal{M}(-D)$ be the $\mathcal{O}_X^\times$ -torsor of local sections $m \in \mathcal{M}$ such that $\alpha_X(m)\mathcal{O}_X = \mathcal{O}_X(-D)$. Let $(A, I, M_{\mathrm{Spf} A})^a$ be a bounded log prism in $(X, \mathcal{M})_\Delta^{\mathrm{op}}$, with structure map

$$f : (\mathrm{Spf} A/I, M_{\mathrm{Spf} A/I}) \rightarrow (X, \mathcal{M}).$$

Let $i : \mathrm{Spf} A/I \rightarrow \mathrm{Spf} A$ be the closed immersion. On $(\mathrm{Spf} A/I)_{\mathrm{ét}}$, set $f^*\mathcal{M}(-D)$ to be obtained from $f^{-1}\mathcal{M}(-D)$ as the $\mathcal{O}_{\mathrm{Spf} A/I}^\times$ -torsor given by change of structure group along $f^{-1}\mathcal{O}_X^\times \rightarrow \mathcal{O}_{\mathrm{Spf} A/I}^\times$. The map $f^b : f^{-1}\mathcal{M} \rightarrow M_{\mathrm{Spf} A/I}$ obtained from the structure map f induces a morphism of étale sheaves $f^*\mathcal{M}(-D) \rightarrow M_{\mathrm{Spf} A/I}$. On $(\mathrm{Spf} A)_{\mathrm{ét}}$, set $\widetilde{\mathcal{M}}_A(-D) := M_{\mathrm{Spf} A} \times_{i_* M_{\mathrm{Spf} A/I}}$

$i_* f^* \mathcal{M}(-D)$. We then finally take

$$\mathcal{O}_\Delta(-D)(A, I, M_{\mathrm{Spf} A}) := \Gamma \left(\mathrm{Spf} A, \mathcal{L}_{\mathcal{O}_{\mathrm{Spf} A}^\times}(\widetilde{\mathcal{M}}_A(-D)) \right),$$

which one can think as of taking an $\mathcal{O}_{\mathrm{Spf} A}$ -span of the generators in $\widetilde{\mathcal{M}}_A(-D)$ we have extracted.

The previous formulation was useful to needed to work globally in an easy manner, but we point out that when working locally one can pick $m_D \in \mathcal{M}$ so $\alpha(m_D)$ is a local defining equation for D . What we called $\mathcal{M}(-D)$ is a globalization of this generated by such sections. Then we have

$$\widetilde{\mathcal{M}}_A(-D) = \left\{ m \in M_A : \mathrm{red}(m) = u f^b(m_D) \text{ locally for } u \in \mathcal{O}_{A/I}^\times \right\}.$$

where $f^b : f^{-1}\mathcal{M} \rightarrow M_{\mathrm{Spf} A/I}$ is obtained from the structure map f and $\mathrm{red} : M_A \rightarrow M_{A/I}$ is the quotient by $1 + I\mathcal{O}_A$.

We can now define $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$.

Definition 5.2. For $n \geq 0$, set

$$\mathcal{O}_\Delta(nD) := \underline{\mathrm{Hom}}_{\mathcal{O}_\Delta} \left(\mathcal{O}_\Delta(-D)^{\otimes n}, \mathcal{O}_\Delta \right).$$

The map $\mathcal{O}_\Delta(-D) \rightarrow \mathcal{O}_\Delta$ gives transition maps $\mathcal{O}_\Delta(nD) \rightarrow \mathcal{O}_\Delta((n+1)D)$. We define

$$\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D) := \left((\mathrm{colim}_{n \geq 0} \mathcal{O}_\Delta(nD))_p^\wedge [1/I_\Delta] \right)_p^\wedge.$$

From this description we also see Frobenius on $\mathcal{O}_\Delta$ extends to $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$. Strict étale locally on a framed log affine, if D is cut out by parameters $T_1, \dots, T_r$, then $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ evaluates on the corresponding log q -de Rham prism as $R_0^\square((q-1))_p^\wedge [1/T_1, \dots, 1/T_r]_p^\wedge$. Lemma 4.30 then computes $T_{\mathrm{ét}}(\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)) \simeq j_! \mathbf{Z}_p$.

As this evaluates to a ring on every log prism, we may consider $\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1}$. This may be described explicitly by a Čech nerve for a framed log affine $(\mathrm{Spf} R, \mathcal{M})$ with framing $\square$. Let $A^{(m)}$ be the Laurentized Čech nerve of the log q -de Rham prism $A^+ = R_0^\square[[q-1]]$. Lemma A.17, and also compatibility of module objects with limits, yields

$$D_\bullet((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)) \simeq \varprojlim_{[m] \in \Delta} D_\bullet(A^{\circ, (m)}).$$

Thus taking dualizable objects and applying an equalizer $(-)^{\varphi=1}$ we obtain

$$\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1} \simeq \varprojlim_{[m] \in \Delta} \mathrm{Perf}(A^{\circ, (m)})^{\varphi=1}.$$

Here, as in Lemma A.16 in the Appendix, we appeal to the fact that (in our situation) one can ignore solid structures when restricting to dualizables.

Lemma 5.3. We have $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D) \in D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$.

Moreover, any crystal in $\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1}$ is also in $D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$.

Proof. For the first claim, the finitely generated unit condition can be checked modulo p by Remark 4.4, and upon doing so the colimit presentation in Definition 5.2 can be rewritten as $(\operatorname{colim}_{n \geq 0} \varphi^{*n} \mathcal{O}_\Delta(D)/p) [1/I_\Delta]$. But then this root presentation makes it clear that after evaluation on any log prism $(A^+, I, \mathcal{M}_{A^+})^a$ the result is in $\operatorname{Thick}(\bar{A}[1/I][F])$ and unit, by the same argument as the footnote in Theorem 4.7.

For the second claim, we use Theorem 4.7 as a shortcut. By Remark 4.15 we can test membership in $D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ using log q -de Rham prisms, so we pick a log q -de Rham prism $(R_0^\square[[q-1]], [p]_q, M)^a$ and wish to show for $A = R_0^\square((q-1))_p^\wedge$ that

$$\operatorname{Perf}(A[1/T_1, \dots, 1/T_r]_p^\wedge)^{\varphi=1} \hookrightarrow D_{\text{fgu}}(A).$$

Again using Remark 4.4, this can be tested modulo p ; then it is clear from the root criterion to test the finitely generated unit condition for a Noetherian regular ring such as $\bar{A} = A/p$.

Indeed, using a Postnikov dévissage (see [Hau24, Lemma 5.4.3]) we reduce to checking the claim on the heart for $M \in \operatorname{Coh}(\bar{A}[1/T_1, \dots, 1/T_r])^{\varphi=1}$. But there it is clear, as we can choose a coherent $\bar{A}$ -submodule $M_0 \subset M$ with $M_0[1/T] = M$, and let $\beta : M \rightarrow \varphi^* M$ be inverse to the unit Frobenius on M . But then for some $N \geq 0$ we have

$$\beta(M_0) \subseteq T^{-N} \varphi^* M_0.$$

It follows that $T^{-k} M_0$ is the desired root once k is large; the claim follows. $\square$

Thus, we can now make sense of the main theorem's statement for this subsection:

Theorem 5.4. *Let $(X, \mathcal{M})$ be as in Assumption 3.1 over $W(k)[\zeta_p]$. Then $T_{\text{ét}}$ fits in a commutative diagram*

$$\begin{array}{ccc} \operatorname{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1, \text{op}} & \xrightarrow{\sim} & D_{\text{lis}}^{(b)}((X_\eta \setminus D_\eta)_{\text{proét}}, \mathbf{Z}_p) \\ \downarrow & & \downarrow j_! \\ D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\text{op}} & \xrightarrow{T_{\text{ét}}} & D_{\text{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbf{Z}_p). \end{array}$$

The top horizontal arrow is an equivalence.

Remark 5.5. Although [BS23, Corollary 3.7] describes the target using ordinary Laurent F-crystals, it is not directly suited to the functor $j_!$. Among several issues is that the complement $X_\eta \setminus D_\eta$ need not be quasicompact; for example, consider $\mathbf{P}_{\mathbf{Q}_p}^1 \setminus \{\infty\} = \mathbf{A}_{\mathbf{Q}_p}^1$. Moreover, a formal model of this complement need not map to the fixed smooth proper model X with generic fiber X_η , and even worse Raynaud models require noncanonical admissible blowups to acquire such a map and need not remain smooth. We fix these problems by using log formal schemes to encode the stratification.

A second remark worth making concerns a comparison to variants of [Yam11] and [Fal89]. As we can access more general constructible sheaves it seems one can likely pursue this further to work out comparison theorems with constructible coefficients, rather than just $j_! \mathbb{L}$ as in the remark.

Remark 5.6. Let $\pi = \zeta_p - 1$ be a uniformizer of $W(k)[\zeta_p]$. If one does not invert I_Δ and instead takes the unitalization of $\mathcal{O}_\Delta(D) \rightarrow \varphi^* \mathcal{O}_\Delta(D)$, one can calculate that the crystalline realization (i.e. the unitalization $\mathcal{E}$ of $\mathcal{O}_{\text{cris}}(D) \rightarrow \varphi^* \mathcal{O}_{\text{cris}}(D)$) satisfies

$$\text{RHom}_{D^\varphi((X, \mathcal{M})_{\pi=0}/W(k))_{\text{cris}}, \mathcal{O}_{\text{cris}}}(\mathcal{E}, \mathcal{O}_{\text{cris}}) \simeq (\text{R}\Gamma_{\text{cris}, c}((X, \mathcal{M})_{\pi=0}/W(k)))^{\varphi=1}.$$

Here compactly supported log crystalline cohomology is as defined in [Tsu99]. This shows the crystalline realization of the natural non-Laurentized variant of $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ is compatible with the expectation one would have from the open C_{st} conjecture.

Before we begin the proof, we must establish some additional basic facts about the category $\text{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1}$.

Lemma 5.7. *Let $(X, \mathcal{M})$ be as in Assumption 3.1 over $W(k)[\zeta_p]$. The category*

$$\text{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1, \text{op}}$$

is a stack for the strict étale topology on such $(X, \mathcal{M})$.

Proof. This follows from Theorem 4.14, because we can extract the desired category

$$\text{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1}$$

as dualizable modules over the idempotent algebra $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ in $D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$. Formally, both of these operations preserve the limit diagram we use to test descent along a strict étale cover $(U, \mathcal{M}_U) \rightarrow (X, \mathcal{M})$, namely that we have a limit diagram

$$D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \simeq \lim_{[n] \in \Delta} D_{\text{fgu}}((U^{(n)}, \mathcal{M}_{U^{(n)}})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$$

for the Čech nerve. This is because $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ is an idempotent algebra in $D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ taking modules over it gives a limit diagram, and then the same holds for taking dualizable objects. This last step is actually not necessary, since the Fitting ideal argument in Proposition 4.28 shows it is redundant. $\square$

Lemma 5.8. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine. Then*

$$\text{Perf}((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1} \simeq \text{Perf}(R_0^\square((q-1))_p^\wedge[1/T_1, \dots, 1/T_r]_p^\wedge)^{\varphi=1, h\Gamma}.$$

Proof. Put $A = R_0^\square((q-1))_p^\wedge$ and $A^\circ = A[1/T_1, \dots, 1/T_r]_p^\wedge$, so that evaluation on the logarithmic q -de Rham prism carries $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ to A° with its Γ -action. Using Lemma 5.3 we can embed the source category into finitely generated unit Laurent crystals, and due to Theorem 4.7 we see $\text{Perf}((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1} \rightarrow \text{Perf}(A^\circ)^{\varphi=1, h\Gamma}$ is fully faithful.

Conversely, let $M \in \text{Perf}(A^\circ)^{\varphi=1, h\Gamma}$. This category clearly sits inside $D_{\text{fgu}}(A)^{h\Gamma}$, thus 4.7 gives an fgu crystal $\mathcal{E}$ with evaluation M . The localization map

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$$

is an equivalence after evaluation on the log q -de Rham prism, hence is an equivalence by conservativity. Thus $\mathcal{E}$ is an $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$ -module. Using the description

$$\mathrm{Perf}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)) \simeq \varprojlim_{[m] \in \Delta} \mathrm{Perf}(A^{\circ, (m)})$$

and that the crystal property identifies evaluation of $\mathcal{E}$ on the m th term of the Čech nerve of the log q -de Rham prism with

$$A^{\circ, (m)} \widehat{\otimes}_{A^\circ}^L M,$$

which is perfect over $A^{\circ, (m)}$, since dualizability is detected termwise in a limit of symmetric monoidal categories $\mathcal{E}$ is perfect over $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D)$. The claim follows. $\square$

We now tackle full faithfulness. Our strategy will be to show this in the perfectoid setting (using the formulation of $T_{\text{ét}}$ in Lemma 4.30). However unlike Laurent F -crystals in perfect complexes, it is not possible to use only the unit Frobenius to reduce to the perfectoid setting. We explain this further in Remark 5.13.

Proposition 5.9. *Let $(\mathrm{Spf} R, \mathcal{M})$ be a framed log affine. The base change functor*

$$\mathrm{Perf}((\mathrm{Spf} R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1} \longrightarrow \mathrm{Perf}(\mathbb{A}_{\text{inf}}((R_0^\square)_\infty)[1/\xi, 1/[T_1^b], \dots, 1/[T_r^b]]_p^\wedge)^{\varphi=1, h\Gamma}$$

is fully faithful.

Proof. As usual, associated to the framed log affine we have a logarithmic q -de Rham prism, so we let $A = R_0^\square((q-1))_p^\wedge$. For this particular proof we only care about a localization, so put $A^\circ := A[1/T_1, \dots, 1/T_r]_p^\wedge$ and $A_{\text{perf}}^\circ = \mathbb{A}_{\text{inf}}((R_0^\square)_\infty)[1/\xi, 1/[T_1^b], \dots, 1/[T_r^b]]_p^\wedge$. We can express the source as $\mathrm{Perf}(A^\circ)^{\varphi=1, h\Gamma}$ using Lemma 5.8; the target is already $\mathrm{Perf}(A_{\text{perf}}^\circ)^{\varphi=1, h\Gamma}$.

By the tensor/Hom adjunction, it suffices to show that

$$(3) \quad R\Gamma_{\text{cont}}(\Gamma, M)^{\varphi=1} \longrightarrow R\Gamma_{\text{cont}}(\Gamma, M \otimes_{A^\circ} A_{\text{perf}}^\circ)^{\varphi=1}$$

is an isomorphism for $M \in \mathrm{Perf}(A^\circ)^{\varphi=1, h\Gamma}$. We can also use the ordinary tensor product $M \otimes_{A^\circ} A_{\text{perf}}^\circ$ in the target because the relevant map $A^\circ \rightarrow A_{\text{perf}}^\circ$ is faithfully flat. Indeed, Kunz's theorem makes $A^+/p \rightarrow A_{\text{perf}}^+/p$ faithfully flat, so $A^+ \rightarrow A_{\text{perf}}^+$ is p -completely faithfully flat. The map $A^\circ \rightarrow A_{\text{perf}}^\circ$ is its p -completed base change after inverting $[p]_q, T_1, \dots, T_r$, so it is p -completely faithfully flat; since A° is Noetherian, it is faithfully flat.

By derived Nakayama we can work with mod p coefficients, which we now do. The source and target both have standard t -structures modulo p obtained by restricting the t -structures on $D(A^\circ/p)$ and $D(A_{\text{perf}}^\circ/p)$. To check these are well-defined one must check truncations remain perfect – this is clear for the source because A°/p is Noetherian regular. For the target the t -structure exists before applying $(-)^{h\Gamma}$, which is implicit in the proof [BS23, Proposition 3.4]: in particular they show that the top cohomology of a perfect unit Frobenius module over an F_p -algebra is a vector bundle. Using a Postnikov dévissage ([Hau24, Lemma 5.4.3]) we reduce to testing the claim for p -torsion M in the heart (which must lie in $\mathrm{Vect}(A^\circ/p)^{\varphi=1, h\Gamma}$).

We now prove (3) for M in $\mathrm{Vect}(A^\circ/p)^{\varphi=1, h\Gamma}$. The faithfully flat map $A^\circ/p \rightarrow A_{\text{perf}}^\circ/p$ is injective, with cokernel $(A_{\text{perf}}^\circ/A^\circ)/p$. Since M is finite projective over A°/p , tensoring this sequence with M is exact. It is therefore enough to prove $R\Gamma_{\text{cont}}(\Gamma, M \otimes_{A^\circ} (A_{\text{perf}}^\circ/A^\circ))^{\varphi=1} = 0$.

Choose $m \gg 0$ and let $(R_0^\square)_m$ be obtained from the framing by adjoining p^m th roots of the coordinates T_i . Define $\varphi^{-m}A^\circ = (R_0^\square)_m((q^{1/p^m} - 1))[1/T_1, \dots, 1/T_r]_p^\wedge$ and $\varphi^{-m}A^{\circ, \text{cyc}} = (R_0^\square)_m((q^{1/p^\infty} - 1))^\wedge[1/T_1, \dots, 1/T_r]_p^\wedge$, viewed as intermediate subrings

$$A^\circ \subset \varphi^{-m}A^\circ \subset \varphi^{-m}A^{\circ, \text{cyc}} \subset A_{\text{perf}}^\circ.$$

Since φ and Γ commute, the unit Frobenius of M gives a Γ -equivariant equivalence

$$R\Gamma_{\text{cont}}(\Gamma, M)^{\varphi=1} \simeq R\Gamma_{\text{cont}}(\Gamma, M \otimes_{A^\circ} \varphi^{-m}A^\circ)^{\varphi=1}.$$

For $m \gg 0$, the exact triangles associated to

$$\varphi^{-m}A^\circ/A^\circ \longrightarrow A_{\text{perf}}^\circ/A^\circ \longrightarrow A_{\text{perf}}^\circ/\varphi^{-m}A^\circ$$

and to $\varphi^{-m}A^{\circ, \text{cyc}}/\varphi^{-m}A^\circ \longrightarrow A_{\text{perf}}^\circ/\varphi^{-m}A^\circ \longrightarrow A_{\text{perf}}^\circ/\varphi^{-m}A^{\circ, \text{cyc}}$ show

$$R\Gamma_{\text{cont}}(\Gamma, M \otimes_{A^\circ} (A_{\text{perf}}^\circ/A^\circ))^{\varphi=1} = 0$$

by applying Lemmas 5.12 and 5.11. The claim then follows. $\square$

We will next provide the needed technical lemmas on continuous Γ cohomology. We retain the notation of Proposition 5.9.

To make sense of the cohomology calculations, we must work out the relevant topologies on A°/p and related rings to compute continuous group cohomology with. Using Remark 4.8 or even Lemma 3.15 (as it still applies here) we see

$$\overline{A}^\circ := A^\circ/p = \bigcup_{i \geq 0} T^{-i}\overline{R}^\square((q-1))$$

is equipped with the inductive limit topology and $T^{-i}\overline{R}^\square((q-1))$ have their topologies induced from $\overline{R}^\square((q-1))$. We must give vector bundles M over this ring their weak topology, which is all we need to deal with. We use similar constructions for $A^{\circ, \text{cyc}}/p$ and A_{perf}°/p .

Let $\overline{A}^+ = (R_0^\square/p)[[q-1]]$. By picking a Γ -stable lattice $M^+ \in \text{Mod}_{\overline{A}^+}$ so $M^+[1/(q-1), 1/T_1, \dots, 1/T_r] = M$, we see just as in the proof of Lemma 3.18 that after shrinking Γ to $\Gamma_0 \leq \Gamma$ that eventually $\gamma_i - 1$ raises the $q-1$ -adic filtration $\text{Fil}^\bullet T^{-\alpha}M^+[1/(q-1)]$ for each multi-index $\alpha \in \mathbf{N}^r$. We specifically can choose $\Gamma_0 = (1 + p^m\mathbf{Z}_p) \ltimes (p^{m-1}\mathbf{Z}_p(1))^n \subset \Gamma$ for $m \gg 0$, which will ultimately be the choice of m we need for Proposition 5.9.

Our strategy to show $R\Gamma(\Gamma, -)$ vanishes in Lemmas 5.12 and 5.11 uses the method from [KL16, Lemmas 5.6.3, 5.6.4]⁸. It appears these results cannot be directly used in what follows, but we emphasize there are not really any new ideas needed for our application.

In the above, what we have been implicitly working with is an Ind-Banach $\mathbf{F}_p$ -module. That is, $M = \text{colim}_{\mathbf{N}} M^{(\mathbf{N})}$, which can be taken in the previous discussion to be $T^{-\mathbf{N}}M^+[1/(q-1)]$. Here $T^{-\mathbf{N}} := T_1^{-N} \cdots T_r^{-N}$, as this is a cofinal subsystem.

We work with a more restrictive variant of Ind-Banach $\mathbf{F}_p$ -modules. We will abbreviate this just to “LB” modules and rings, but our convention is the following. An LB module is a presentation

⁸It also appears there are some gaps in this section, but we emphasize we do not directly use any results from their article (only the general method).

$M = \text{colim}_{N \in \mathbb{N}} M^{(N)}$ by Banach modules with closed contractive embeddings. LB rings are algebra objects with LB underlying modules, and group actions satisfy $h \cdot M^{(N)} = M^{(N)}$. We write

$$\widehat{\bigoplus_j}^{\text{LB}} M_j := \text{colim}_N \widehat{\bigoplus_j} M_j^{(N)},$$

where the completed sum is taken for the supremum norm. In the applications, all norms are normalized by the same value of $|q - 1|$.

For a closed subgroup $\Gamma_0 \leq \Gamma$, choose topological generators $g_1, \dots, g_k$ of $\Gamma_0 \cap \mathbf{Z}_p(1)^n \simeq \mathbf{Z}_p^k$. If the image of Γ_0 in $1 + p\mathbf{Z}_p$ is nontrivial, choose $g_0 \in \Gamma_0$ lifting a topological generator of this image. Using the usual resolution for group cohomology of $\mathbf{Z}_p^k$ and Hochschild-Serre we may compute $R\Gamma_{\text{cont}}(\Gamma_0, M)$ as the fiber of $g_0 - 1$ on

$$\text{Kos}^\bullet(M; g_1 - 1, \dots, g_k - 1),$$

where g_0 acts on the coefficients and by conjugation on the Koszul generators. If the image in $1 + p\mathbf{Z}_p$ is trivial, the Koszul complex itself computes cohomology.

This cochain model of cohomology allows us to make the following definitions, subject to the choices of generators above: a complex is *strictly contractible* if it admits a continuous contracting homotopy. For a family of complexes $C_j^\bullet$, such homotopies h_j^i are *uniformly bounded* if they preserve the $C_j^{i, (N)}$ and

$$\sup_{j,i} \|h_j^i : C_j^{i, (N)} \longrightarrow C_j^{i-1, (N)}\| < \infty$$

for every N .

The following simple observation will be used throughout.

Lemma 5.10. *Let $\Gamma_0 \leq \Gamma$ be a closed subgroup and let B be an LB ring with a continuous action of Γ_0 , and let V be an LB B -module with a continuous Γ_0 -action.*

Suppose that we have a Γ_0 -equivariant decomposition of LB modules

$$\widehat{\bigoplus_{j \geq 0}}^{\text{LB}} V_j \xrightarrow{\sim} V,$$

such that the cochain complexes $R\Gamma(\Gamma_0, V_j)$ admit uniformly bounded contracting homotopies. Then $R\Gamma(\Gamma_0, V) = 0$.

Proof. This is immediate from the definitions: uniform boundedness extends the contracting homotopy to $R\Gamma(\Gamma_0, V)$. $\square$

Note here we implicitly have made a choice of generators to produce the cochain models for cohomology.

Lemma 5.11. *Let $M \in \text{Vect}(A^\circ/p)^{\varphi=1, h\Gamma}$. For $m \gg 0$,*

$$R\Gamma_{\text{cont}}(\Gamma, M \widehat{\otimes}_{A^\circ} (\varphi^{-m} A^{\circ, \text{cyc}} / \varphi^{-m} A^\circ)) = 0.$$

Proof. It suffices to show the cohomology for $\Gamma_0 = (1 + p^m \mathbf{Z}_p) \ltimes (p^{m-1} \mathbf{Z}_p(1))^n$ vanishes, where m is chosen as before so that $\gamma_i - 1$ raise the $(q-1)$ -adic filtration. We call the normal subgroup $H = (p^{m-1} \mathbf{Z}_p(1))^n$.

To apply Lemma 5.10, we will decompose $M \otimes_{\bar{A}^\circ} (\varphi^{-m} \bar{A}^{\circ, \text{cyc}} / \varphi^{-m} \bar{A}^\circ)$. To do this, we set $B_0 = \varphi^{-m} \bar{A}^\circ = (\bar{R}_0^\square)_m((q^{1/p^m} - 1))[1/T_1, \dots, 1/T_r]$. Recall we used the notation $(\bar{R}_0^\square)_m$ to denote that we have adjoined $1/p^m$ th powers of the T_i . We then set $B_j = (\bar{R}_0^\square)_m((q^{1/p^{m+j}} - 1))[1/T_1, \dots, 1/T_r]$, so B_∞ is $\varphi^{-m} \bar{A}^{\circ, \text{cyc}}$. We write M_{B_j} for the base change of M to one of these LB rings.

Then

$$B_{j+1} = \bigoplus_{a=0}^{p-1} B_j q^{a/p^{m+j+1}}$$

allows us to define Γ -equivariant B_j -linear retractions ψ_j by projection onto $a = 0$. With the prescribed LB structures,

$$\widehat{\bigoplus_j^{\text{LB}} \ker(\psi_j)} \simeq B_\infty / B_0.$$

Tensoring up with the vector bundle M , we get a decomposition by $M_{B_j} \widehat{\otimes} \ker \psi_j$ of $M \otimes_{\bar{A}^\circ} (\varphi^{-m} \bar{A}^{\circ, \text{cyc}} / \varphi^{-m} \bar{A}^\circ) = M_{B_\infty} / M_{B_0}$.

Thus it only remains to produce the contracting homotopies. This is tricky, because our Γ_0 is not commutative. However Lemma 5.10 still applies here, as we still have produced an explicit cochain model for Γ_0 -cohomology in this case. For this model, we recall we had to fix generators. We fix $\gamma_0 = 1 + p^m$ generating $1 + p^m \mathbf{Z}_p$, and γ_i generators of $H = (p^{m-1} \mathbf{Z}_p(1))^n$.

It suffices actually to produce a contracting homotopy for the fiber of $\gamma_0 - 1$, but for the entire Koszul complex $\text{Kos}_H^\bullet(M_{B_j} \otimes_{B_j} \ker \psi_j)$. Using the aforementioned decompositions $B_{j+1} = \bigoplus_{a=0}^{p-1} B_j q^{a/p^{m+j+1}}$, since the decomposition is H -equivariant we can decompose further into

$$\text{Kos}_H^\bullet(M_{B_j} \otimes_{B_j} \ker \psi_j) \simeq \bigoplus_{1 \leq a \leq p-1} \text{Kos}_H^\bullet(M_{B_j} q^{a/p^{m+j+1}}).$$

To apply Lemma 5.10 it will suffice to produce a contracting homotopy for the action of $\gamma_0 - 1$ on these summands.

For each a , the summands $\text{Kos}_H^\bullet(M_{B_j} q^{a/p^{m+j+1}})$ in question can be written as

$$M_{B_j} q^{a/p^{m+j+1}} \longrightarrow M_{B_j} q^{a/p^{m+j+1}} \otimes_{\mathbf{Z}_p} \Lambda^1(H) \longrightarrow \dots \longrightarrow M_{B_j} q^{a/p^{m+j+1}} \otimes_{\mathbf{Z}_p} \Lambda^n(H).$$

For $I \subseteq \{1, \dots, n\}$, write $e_I = \bigwedge_{i \in I} e_i$ for the corresponding basis element of the term for $\Lambda^{|I|}(H)$. Using $\gamma_0 \gamma_i = \gamma_i^{1+p^m} \gamma_0$ we see

$$\gamma_0(- \cdot e_I) = \left(\prod_{i \in I} \frac{\gamma_i^{1+p^m} - 1}{\gamma_i - 1} \right)^{-1} \gamma_0(-) e_I.$$

On the summand $M_{B_j} q^{a/p^{m+j+1}}$ we have $\gamma_0 - 1 = (q^{a/p^{j+1}} - 1) \left[1 + (q^{a/p^{j+1}} - 1)^{-1} (\gamma_0 - q^{a/p^{j+1}}) \right]$.

Thus to give a contracting homotopy on the mapping fiber, it suffices to show the bound $\|(q^{a/p^{j+1}} - 1)^{-1} (\gamma_0 - q^{a/p^{j+1}})\|_N < 1$. We actually do not care if the bound depends on j , since in computing

$\|(\gamma_0 - 1)^{-1}\|_N$ the inverse of $\left[1 + (q^{a/p^{j+1}} - 1)^{-1}(\gamma_0 - q^{a/p^{j+1}})\right]$ has norm ≤ 1 – thus the bound is entirely dependent on $|(q^{a/p^{j+1}} - 1)|^{-1} < |q - 1|^{-1}$, making it uniform.

By using the action on B_j and using that $\gamma_0 - 1$ raises the $q - 1$ -adic filtration for coefficients this action eventually satisfies for $m \gg 0$

$$\left\| (q^{a/p^{j+1}} - 1)^{-1}(\gamma_0 - q^{a/p^{j+1}}) \right\|_N \leq |q - 1|^{p^{-j}(1-p^{-m}-p^{-1})} < 1.$$

This handles the fiber of $\gamma_0 - 1$ on $M_{B_j} q^{a/p^{m+j+1}}$. For the full Koszul complex, note that there are finitely many $I \subseteq \{1, \dots, n\}$ to deal with and $\left\| \left(\prod_{i \in I} \frac{\gamma_i^{1+p^m} - 1}{\gamma_i - 1} \right)^{-1} - 1 \right\|_N \leq |q - 1|^{p^{m-1}}$ (again crucially using that $\gamma_i - 1$ raise the $q - 1$ -adic filtration).

Thus we have produced the required homotopies and Lemma 5.10 finishes the claim. Note that we only showed vanishing Γ_0 cohomology, but by Hochschild-Serre this proves there is no Γ -cohomology as well. $\square$

Fortunately, the remaining lemma is easier because we can deal with an abelian group. The proof is nearly identical but we explain it fully for completeness.

Lemma 5.12. *Let $M \in \text{Vect}(A^\circ/p)^{\varphi=1, h\Gamma}$. For $m \gg 0$,*

$$R\Gamma_{\text{cont}}(\Gamma, M \widehat{\otimes}_{A^\circ} (A_{\text{perf}}^\circ / \varphi^{-m} A^{\circ, \text{cyc}})) = 0.$$

Proof. The argument is very similar, so we simply indicate the decomposition we use and contracting homotopies to apply the same strategy as the previous lemma.

In this case we put $B_j = \varphi^{-(m+j)} A^{\circ, \text{cyc}} / p$ and write M_{B_j} for the base changes to these rings, and $\Gamma_0 = (p^{m-1} \mathbf{Z}_p(1))^n$. We must produce a Γ_0 -equivariant decomposition of

$$M \otimes_{A^\circ} (A_{\text{perf}}^\circ / \varphi^{-m} A^{\circ, \text{cyc}}) = M_{B_\infty} / M_{B_0}.$$

Now for $\alpha = (a_1, \dots, a_n)$, there is a decomposition

$$B_{j+1} = \bigoplus_{\alpha \in \{0, \dots, p-1\}^n} B_j \prod_{\nu=1}^n T_\nu^{a_\nu / p^{m+j+1}}$$

This defines ψ_j as the projection onto the $\alpha = \vec{0}$ component. With the obvious LB structures, these give

$$\widehat{\bigoplus_{j \geq 0}^{\text{LB}}} \ker(\psi_j) \simeq B_\infty / B_0,$$

which we can similarly tensor with M to get a decomposition of M_{B_∞} / M_{B_0} . We must then produce contracting homotopies for the individual summands $M_{B_j} \widehat{\otimes} \ker \psi_j$.

Using the generators $\gamma_1, \dots, \gamma_n$ of Γ_0 , the preceding Γ_0 -equivariant decomposition reduces us to contracting the complexes

$$\text{Kos}_{\Gamma_0}^\bullet \left(M_{B_j} \prod_{\nu} T_\nu^{a_\nu / p^{m+j+1}} \right)$$

for $\alpha \neq 0$. Choose the least i with $a_i \neq 0$. We have

$$\gamma_i - 1 = (q^{a_i/p^{j+1}} - 1) \left[1 + (q^{a_i/p^{j+1}} - 1)^{-1} (\gamma_i - q^{a_i/p^{j+1}}) \right].$$

To contract the fiber of $\gamma_i - 1$ uniformly, as before it suffices to show

$$\left\| (q^{a_i/p^{j+1}} - 1)^{-1} (\gamma_i - q^{a_i/p^{j+1}}) \right\|_{\mathbb{N}} < 1.$$

By using the action on B_j and that $\gamma_i - 1$ is $q - 1$ -adic filtration raising for coefficients (by our choice of $m \gg 0$) we obtain

$$\left\| (q^{a_i/p^{j+1}} - 1)^{-1} (\gamma_i - q^{a_i/p^{j+1}}) \right\|_{\mathbb{N}} \leq |q - 1|^{(p-1)p^{-j-1}} < 1.$$

This gives $(\gamma_i - 1)^{-1}$ of norm at most $|q - 1|^{-1}$ on each of the summands for M_{B_j} where $\alpha \neq 0$. For the full Koszul complex, we can use a similar argument to the previous lemma for $\gamma_i - 1$. These homotopies are compatible and have the same bound, so lemma 5.10 then gives the vanishing for continuous Γ_0 cohomology, and Hochschild-Serre gives it for Γ . $\square$

Remark 5.13. It is tempting to try to argue that Lemma 5.11 would only need to use φ , as this is the case for ordinary (φ, Γ) modules using invariance of étale φ -modules under completed perfection. Consider the map

$$A = k((q - 1))\langle T \rangle[1/T] \rightarrow B = k((q^{1/p^\infty} - 1))^{\wedge} \langle (T^b)^{1/p^\infty} \rangle[1/T^b],$$

with $T \mapsto T^b$. If full faithfulness followed just from invariance under this completed perfection, then $\text{Perf}(A)^{\varphi=1} \rightarrow \text{Perf}(B)^{\varphi=1}$ would be fully faithful. By Katz's theorem this would force $H_{\text{ét}}^1(A, \mathbf{F}_p) \rightarrow H_{\text{ét}}^1(B, \mathbf{F}_p)$ to be an isomorphism. One can check that the Artin-Schreier class on B defined by

$$f = \sum_{n \geq 1} (q - 1)^n (T^b)^{-1+1/p^n}$$

is not pulled back from A , so the étale cohomology groups are not isomorphic via the natural base change map. The issue here is roughly that passage to B mixes completed and decompleted perfection.

Proposition 5.14. *The top horizontal functor in Theorem 5.4 is fully faithful.*

Proof. Lemma 5.7 and strict étale descent for $D_{\text{lisse}}^{(b)}((X_\eta \setminus D_\eta)_{\text{proét}}, \mathbf{Z}_p)$, together with Lemma 3.4, reduce the claim to a framed log affine. Lemma 5.8 identifies the source with $\text{Perf}(R_0^\square((q - 1))_p^{\wedge}[1/T_1, \dots, 1/T_r]_p^{\wedge})^{\varphi=1, h\Gamma}$, and Proposition 5.9 makes base change to

$$\mathbb{A}_{\text{inf}}((R_0^\square)_\infty)[1/\xi, 1/[T_1^b], \dots, 1/[T_r^b]]_p^{\wedge}$$

fully faithful. Lemma 4.30 supplies a formula for $T_{\text{ét}}$ where this is the first step, and the remaining steps are obviously fully faithful (for example tilting or Γ -descent) with the exception of analytification.

Thus we must investigate full faithfulness of analytification. Put

$$U = \text{Spec}(R_0^\square)_{\infty, \eta}^b[1/T_1^b, \dots, 1/T_r^b].$$

For $\mathbb{L}, \mathbb{L}' \in D_{\text{lis}}^{(b)}(U_{\text{ét}}, \mathbf{F}_p)$, the monodromy of each $H^i \underline{\text{RHom}}(\mathbb{L}, \mathbb{L}')$ is contained in a finite group $\text{GL}_{m_i}(\mathbf{F}_p)$. Boundedness gives one finite étale cover $U' \rightarrow U$ on which $H^i \underline{\text{RHom}}(\mathbb{L}, \mathbb{L}') \simeq \mathbf{F}_p^{m_i}$ for every i . On U' , [Bha25, Remark 6.2.7] further gives

$$\text{R}\Gamma(U', \mathbf{F}_p) \xrightarrow{\sim} \text{R}\Gamma(U'^{\text{an}}, \mathbf{F}_p).$$

Finally, a Postnikov dévissage combined with finite étale descent to trivialize coefficients to $\mathbf{F}_p$ gives full faithfulness modulo p ; derived Nakayama then gives full faithfulness of analytification. $\square$

We next prove essential surjectivity after passing, on a framed strict étale chart, to a finite prime-to- p Kummer cover. Proposition 5.15 descends the object constructed on the cover, using the full faithfulness already proved.

Proposition 5.15. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine with coordinates $T_1, \dots, T_r$, and let N be prime to p . Put*

$$R_N = R\langle S_1, \dots, S_r \rangle / (S_1^N - T_1, \dots, S_r^N - T_r),$$

and equip $\text{Spf } R_N$ with the log structure $\mathcal{M}_N$ induced by $V(S_1 \cdots S_r)$. Then

$$\left(\text{Perf}((\text{Spf } R_N, \mathcal{M}_N)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D_N))^{\varphi=1, \text{op}} \right)^{h\mu_N^r} \simeq \text{Perf}((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D))^{\varphi=1, \text{op}}.$$

Proof. Let $f_N : (\text{Spf } R_N, \mathcal{M}_N) \rightarrow (\text{Spf } R, \mathcal{M})$ be natural map of log formal schemes, where as above $R_N = R\langle S_1, \dots, S_r \rangle / (S_1^N - T_1, \dots, S_r^N - T_r)$. Since $f_N^*D = ND_N$, there is a canonical equivalence

$$f_N^* \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D) \simeq \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*ND_N) \simeq \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D_N).$$

Thus pullback induces a functor on the corresponding module categories

$$\text{Perf}((\text{Spf } R, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D))^{\varphi=1, \text{op}} \longrightarrow \text{Perf}((\text{Spf } R_N, \mathcal{M}_N)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D_N))^{\varphi=1, \text{op}}.$$

we will use to induce the desired equivalence.

The claim becomes more transparent when we write out the Čech nerves for the source and target. Base changing the framing for $(\text{Spf } R, \mathcal{M})$ along $T_i \mapsto S_i^N$ gives $(\text{Spf } R_N, \mathcal{M}_N)$ a framing $\square_N$, so it also has an associated q -de Rham prism. Let $A = R_0^\square((q-1))_p^\wedge$, and $A_N = R_{0,N}^\square((q-1))_p^\wedge$ be their corresponding (Laurentized) log q -de Rham prisms. As usual we write $A^{(s)}$ and $A_N^{(s)}$ for the Laurentized Čech nerves. We write for $A^{(s)}(*D)$ and $A_N^{(s)}(*D)$ for the evaluations of $\mathcal{O}_\Delta[1/I_\Delta]^\wedge(*D)$ on these prisms.

We then obtain the following diagram:

$$\begin{array}{ccccccc} \text{Perf}(A_N(*D_N))^{\varphi=1} & \rightrightarrows & \text{Perf}(A_N^{(1)}(*D_N))^{\varphi=1} & \rightrightarrows & \text{Perf}(A_N^{(2)}(*D_N))^{\varphi=1} & \rightrightarrows & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Perf}(A(*D))^{\varphi=1} & \rightrightarrows & \text{Perf}(A^{(1)}(*D))^{\varphi=1} & \rightrightarrows & \text{Perf}(A^{(2)}(*D))^{\varphi=1} & \rightrightarrows & \dots \end{array}$$

The rings on top all have a grading by $\alpha \in (\mathbf{Z}/N\mathbf{Z})^r$, making Perf acquire an action of μ_N^r ⁹. For example, $R_{0,N}^\square = R_0^\square \langle S_1, \dots, S_r \rangle / (S_1^N - T_1, \dots, S_r^N - T_r)$, so

$$A_N(*D) = \bigoplus_{\alpha \in (\mathbf{Z}/N\mathbf{Z})^r} A(*D) \cdot S^\alpha,$$

with the $\alpha = 0$ component being the natural inclusion of $A(*D)$. For the first Čech term, the analogous grading induced from the one on $R_{0,N}^\square$ again holds, even at the level of $A_N^{+, (1)}$ and $A^{+, (1)}$. We see this by noting that the elements $v_i = S_{i,1}/S_{i,0}$ we adjoin (in the notation of Lemma 3.6; for $A_N^{+, (1)}$ we use v_i rather than u_i to distinguish them) satisfies $v_i = (S_{i,1}/S_{i,0}) = (T_{i,1}/T_{i,0})^{1/N} = u_i^{1/N} \in A^{+, (1)}$ since N is prime to p . Indeed

$$u_i^{1/N} = \sum_{k \geq 0} \binom{1/N}{k} (u_i - 1)^k,$$

which converges $(p, [p]_q)$ -adically since we force $u_i - 1 \in [p]_q A^{+, (1)}$. For higher Čech terms, the claim is clear because they are obtained from iterated tensor products of $A^{+, (1)}$ over A^+ (and similarly for A_N).

Even better, here

$$A^{(s)}(*D) \longrightarrow A_N^{(s)}(*D_N)$$

is a finite étale μ_N^r torsor. Indeed, since we have p -completely inverted $T_1, \dots, T_r$ and $S_1, \dots, S_r$ the map becomes finite étale.

It therefore follows that $\text{Perf}(A^{(s)}(*D))^{\varphi=1} \simeq \text{Perf}(A_N^{(s)}(*D_N))^{\varphi=1, h\mu_N^r}$ by finite étale descent, noting that the μ_N^r action commutes with φ . The claim then follows as these equivalences are compatible with the maps in the Čech nerve. $\square$

We note that concretely one may descend a crystal down by taking the projection into $\alpha = 0$ components, which is indeed Γ -equivariant. We will use this shortly.

Corollary 5.16. *The top horizontal functor in Theorem 5.4 is an equivalence.*

Proof. By Proposition 5.14, it only remains to prove essential surjectivity. Since X_η is quasicompact and smooth, any $\mathbb{L} \in D_{\text{lis}}^{(b)}((X_\eta \setminus D_\eta)_{\text{proét}}, \mathbf{Z}_p)$ is actually globally bounded rather than simply locally bounded. Thus for the purpose of showing essential surjectivity, since $\mathbb{L} \in D_{\text{lis}}^{(b)}((X_\eta \setminus D_\eta)_{\text{proét}}, \mathbf{Z}_p)^\heartsuit$ generate bounded objects under shifts and fibers and the functor is exact we only need essential surjectivity on the heart. Moreover, by strict étale descent we may also reduce to testing the equivalence when $(X, \mathcal{M})$ is a framed log affine.

By the previous discussion it will then suffice to directly construct a preimage

$$\mathcal{E} \in \text{Coh}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D))^{\varphi=1, \text{op}}$$

so for such an $\mathbb{L}$ we get $T_{\text{ét}}(\mathcal{E}) = j_! \mathbb{L}$. By [DLLZ23, Corollary 6.3.4], $\mathbb{L}^{\text{ext}} := j_{\text{két},*} \mathbb{L}$ is a Kummer-étale $\mathbf{Z}_p$ -local system on the log generic fiber. We want to argue there is a single prime-to- p Kummer cover $f_N : (X_N, \mathcal{M}_N) \rightarrow (X, \mathcal{M})$ as in Proposition 5.15 such that $f_N^* \mathbb{L}^{\text{ext}}$ is p -Kummer. Without loss of generality we assume $(\text{Spf } R, \mathcal{M})_\eta$ is connected, as we can take the least common

⁹We also need the condition $R_\alpha \cdot R_\beta \subseteq R_{\alpha+\beta}$ for $\alpha, \beta \in (\mathbf{Z}/N\mathbf{Z})^r$ for a general ring R , which is satisfied here.

multiple of such N on each component. By [IKY26, Definition 4] we want to show that if $\bar{x}$ is a log geometric point of $(\mathrm{Spf} R, \mathcal{M})_\eta$ and $(\mathrm{Spf} R, \mathcal{M})_{\eta, \bar{x}}$ denotes strict localization (at the underlying geometric point) with the induced log structure then

$$\pi_1^{\mathrm{k\acute{e}t}}((\mathrm{Spf} R, \mathcal{M})_{\eta, \bar{x}}, \bar{x}) \xrightarrow{\rho} \mathrm{Aut}_{\mathbf{Z}_p}(\mathbb{L}_{\bar{x}}^{\mathrm{ext}})$$

has pro- p image after pullback along f_N for some N . Here ρ is the associated representation to $\mathbb{L}^{\mathrm{ext}}$. The key observation is that $\mathrm{Aut}_{\mathbf{Z}_p}(\mathbb{L}_{\bar{x}}^{\mathrm{ext}}) \rightarrow \mathrm{GL}_n(\mathbf{F}_p)$ has pro- p kernel, where n is the rank of the mod p reduction of $\mathbb{L}$ (which makes sense by connectedness). Thus to obtain a pro- p image we can take N to be the prime-to- p part of $|\mathrm{GL}_n(\mathbf{F}_p)|$, as the effect of pullback along f_N takes N th powers in the image (see the proof of Proposition 5 in [IKY26]).

Thus we may assume $f_N^* \mathbb{L}^{\mathrm{ext}}$ is p -Kummer. By the contravariant version of [IKY26, Theorem 7] we have an equivalence

$$\mathrm{Perf}((X_N, \mathcal{M}_N)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\varphi=1, \mathrm{op}} \simeq D_{\mathrm{lis\acute{e}}}^{(b), \mathrm{pKum}}((X_N, \mathcal{M}_N)_\eta, \mathbf{Z}_p),$$

therefore $f_N^* \mathbb{L}^{\mathrm{ext}}$ has a corresponding perfect Laurent F-crystal $\mathcal{E}$ on $(X_N, \mathcal{M}_N)_\Delta$, or that $T_{\acute{e}t}(\mathcal{E}) \simeq f_N^* \mathbb{L}^{\mathrm{ext}}$.

Let $j_N : X_{N, \eta} \setminus D_{N, \eta} \hookrightarrow (X_N, \mathcal{M}_N)_\eta$ denote the inclusion. Since $T_{\acute{e}t}(\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D_N)) \simeq j_{N,!} \mathbf{Z}_p$ using Lemma 4.30 and $T_{\acute{e}t}$ is symmetric monoidal (from Remark 4.31) we deduce

$$T_{\acute{e}t}(\mathcal{E} \otimes \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D_N)) \simeq j_{N,!} f_N^* \mathbb{L}.$$

On the p -Kummer-étale side of the Koshikawa-Yao correspondence, $f_N^* \mathbb{L}^{\mathrm{ext}}$ by descent comes equipped with an action of μ_N^r , which then extends to $\mathcal{E} \otimes \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D_N)$ (noting $\mathcal{O}_\Delta[1/I_\Delta]_p^\wedge(*D_N)$ has its own μ_N^r -action). Using Proposition 5.15 gives a descended crystal that realizes $j_! \mathbb{L}$, which we know by virtue of having a μ_N^r -equivariant map upstairs and pullback compatibility. $\square$

5.2. Proof of the main theorem. By this point, we have seen that both the source and target of

$$T_{\acute{e}t} : D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\mathrm{op}} \longrightarrow D_{\mathrm{cons}, D}^{(b), \mathrm{pKum}}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$$

satisfy descent for strict étale maps. Therefore we will localize to the situation of a framed log affine for the majority of this section, using Lemma 3.4.

Additionally we now know that there is an equivalence on the open stratum, and that the functor lands in the correct target by Proposition 4.32. The strategy we will carry out here is to match thick generators of both sides, using the following standard lemma.

Lemma 5.17. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between idempotent complete stable ∞ -categories which are thickly generated by objects $\{C_i\}_{i \in I}$ and $\{D_i\}_{i \in I}$, such that $F(C_i) = D_i$ and the morphism*

$$\mathrm{RHom}(C_i, C_j) \longrightarrow \mathrm{RHom}(D_i, D_j)$$

induced by F is an equivalence for any $i, j \in I$. Then F is an equivalence.

Proof. The objects $C \in \mathcal{C}$ such that $\mathrm{RHom}(C_i, C) \longrightarrow \mathrm{RHom}(D_i, F(C))$ is an equivalence form a thick subcategory containing the C_i , hence all of $\mathcal{C}$. Similarly we can run this argument in the other variable, deducing F is fully faithful. Showing essential surjectivity is then straightforward, as the essential image is a thick subcategory containing the D_i hence all of $\mathcal{D}$. Here we have used idempotent completeness of $\mathcal{C}$ to ensure the essential image is closed under retracts. $\square$

In our situation with $T_{\text{ét}}$ on a framed log affine, it is easy to see all the hypotheses are satisfied. Idempotent completeness is clear on the source (which is actually all we used in the proof!) by the description in Theorem 4.7. We can also see it directly, because $D_{\blacksquare}((\text{Spf } R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$ is idempotent complete and the finitely generated unit definition is, by construction, closed under retracts as we ask for membership in $\text{Thick}(A[1/I]_p^{\wedge}\langle F \rangle)$ for every log prism.

The following toy model explains a little more about the architecture of the proof, using Lemma 5.17.

A toy model. Consider the disk $\mathbb{D}_{\mathbf{Q}_p(\zeta_p)}$ punctured at the origin; we use the coordinate T so $\mathbb{D} = \text{Spa } \mathbf{Q}_p(\zeta_p)\langle T \rangle$, equipped with the log structure $\mathcal{M}_{V(T)}$. We use the usual formal model $(\text{Spf } \mathbf{Z}_p[\zeta_p]\langle T \rangle, \mathcal{M}_{V(T)})$.

Let $j : \mathbb{D}^{\times} \hookrightarrow \mathbb{D}$ and $i : V(T) \hookrightarrow \mathbb{D}$ denote the two strata. If $A = \mathbf{Z}_p\langle T \rangle((q-1))_p^{\wedge}$ and $\Gamma = (1 + p\mathbf{Z}_p) \ltimes \mathbf{Z}_p(1)$, the setup of the argument is summarized by the diagram

$$\begin{array}{ccccc}
 \left\{ \begin{array}{c} \text{open stratum unit} \\ \text{Laurent crystals} \end{array} \right\}^{\text{op}} & \xrightarrow{\quad} & \left\{ \begin{array}{c} \text{fgu Laurent crystals} \\ \text{on } (\text{Spf } \mathbf{Z}_p[\zeta_p]\langle T \rangle, \mathcal{M}_{V(T)})_{\Delta} \end{array} \right\}^{\text{op}} & \xrightarrow{\langle i_+(-) \rangle [-1]} & (D_{\text{fgu}}(A/(T))^{h\Gamma})^{\text{op}} \\
 \downarrow T_{\text{ét}} & & \downarrow T_{\text{ét}} & & \downarrow T_{\text{ét}} \\
 \left\{ \begin{array}{c} p\text{-Kummer local systems} \\ \text{on } \mathbb{D}^{\times} \end{array} \right\} & \xrightarrow{j_!} & \left\{ \begin{array}{c} p\text{-Kummer constructible sheaves} \\ \text{on } (\mathbb{D}, \mathcal{M}_{V(T)}) \end{array} \right\} & \xleftarrow{i_*} & \left\{ \begin{array}{c} p\text{-Kummer local systems} \\ \text{on } (V(T), i^* \mathcal{M}_{V(T)}) \end{array} \right\}.
 \end{array}$$

The left vertical arrow is an equivalence by Theorem 5.4, and tensoring with $\mathcal{O}_{\Delta}(*V(T))$ identifies with $j_!$ under $T_{\text{ét}}$. The factor of Γ corresponding to T acts trivially on $A/(T)$, but its action on objects records the normal Kummer inertia on the logarithmic point. The dashed arrow is supplied by Lemma 5.19, or Kashiwara's lemma.

We will ultimately also end up checking that the right vertical arrow is also an equivalence. In this particular case, it is substantially easier than the open stratum result because it is a hollow log point: one can run the usual proof of the (φ, Γ) -module equivalence for a point, except with the perfectoid cover $(\text{Spf } \mathbf{Z}_p\langle \zeta_{p^{\infty}} \rangle)_{\eta}$ we must equip it with the hollow log structure $\mathbf{N}[1/p] \rightarrow 0$ to be a pro-Kummer-étale Γ torsor over a hollow log point. Running a tilting and deperfection argument (noting the analogue of the p -Kummer criterion using Lemma 4.25 holds here) one obtains the equivalence.

One must then show that on either the top or bottom row, objects coming from the hollow point or open stratum thickly generate. For the bottom row the strategy is to use localization triangles

$$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F}$$

and on the top row we will eventually do the same. Thus to use Lemma 5.17, we only need the RHom equivalences. There are no maps from sheaves supported on the open stratum to the closed stratum, and analogously on the finitely generated unit module side, so apart from the equivalences we already know the only point to check is whether or we have full faithfulness for maps

$$\mathcal{E}_{\mathbb{D}^{\times}} \longrightarrow \mathcal{E}_{V(T)},$$

with the arrow reversed after applying $T_{\text{ét}}$. Here the source is an open stratum Laurent crystal and the target in $D_{\text{fgu}}(A/(T))^{h\Gamma}$.

Put $j_!\mathbb{L} = T_{\text{ét}}(\mathcal{E}_{\mathbb{D}^\times})$ and $i_*\mathbb{L}' = T_{\text{ét}}(\mathcal{E}'_{V(T)})$. We will show that we may assume without loss of generality $\mathbb{L}$ extends across the origin to the p -Kummer derived local system $Rj_*\mathbb{L}$: after replacing the disk by a prime-to- p Kummer cover $\mathbb{D}_N$ this will be true, and ultimately we can descend the full faithfulness conclusion from this case.

Thus we assume there is

$$\mathcal{E}_{\mathbb{D}} \in \text{Perf}((\text{Spf } \mathbf{Z}_p[\zeta_p]\langle T \rangle, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\varphi=1}$$

such that $T_{\text{ét}}(\mathcal{E}_{\mathbb{D}}) \simeq Rj_*\mathbb{L}$ and $\mathcal{E}_{\mathbb{D}^\times} \simeq \mathcal{E}_{\mathbb{D}}(*V(T))$. On the Kummer-étale side, one has

$$\text{RHom}(i_*\mathbb{L}', Rj_*\mathbb{L}) = 0.$$

Indeed, the adjunction between j^* and Rj_* identifies this with $\text{RHom}(j^*i_*\mathbb{L}', \mathbb{L}) = 0$. Applying $\text{RHom}(i_*\mathbb{L}', -)$ to the triangle

$$j_!\mathbb{L} \longrightarrow Rj_*\mathbb{L} \longrightarrow i_*i^*Rj_*\mathbb{L}$$

and using the adjunction between i^* and i_* gives $\text{RHom}(i_*\mathbb{L}', j_!\mathbb{L}) \simeq \text{RHom}(\mathbb{L}', i^*Rj_*\mathbb{L})[-1]$. This now consists of sheaves supported at the origin $V(T) \subset \mathbb{D}$, so producing an analogous isomorphism on the crystal side would reduce the needed result to a hollow point where the claim is known. The general proof of the theorem will proceed along similar lines to this toy model, but when there are multiple irreducible components the combinatorics is more complicated.

The general proof. We now observe there is a natural class of thick generators for the target category that we already know. In the situation of a framed log affine $(\text{Spf } R, \mathcal{M}) \xrightarrow{\square} \mathbb{T}_r^n$ we have log strata for $\Lambda \subseteq \{1, \dots, r\}$

$$(\text{Spf } R, \mathcal{M})_{\Lambda, \eta} = \bigcap_{\lambda \in \Lambda} D_{\lambda, \eta} \setminus \bigcup_{\lambda \notin \Lambda} D_{\lambda, \eta}$$

where the open stratum corresponds to the empty set. Here D_{λ} is the zero locus of the étale lift of T_{λ} , or one of the irreducible components of the divisor. The following lemma is hopefully clear to the reader.

Lemma 5.18. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine. The category $D_{\text{cons}, D}^{(b), p\text{Kum}}((\text{Spf } R, \mathcal{M})_{\eta}, \mathbf{Z}_p)$ is thickly generated by $j_{\Lambda, !}\mathbb{L}$ for $\mathbb{L}$ a p -Kummer sheaf in $D_{\text{lis}}^{(b)}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p)$, where*

$$j_{\Lambda} : (\text{Spf } R, \mathcal{M})_{\Lambda, \eta} \hookrightarrow (X, \mathcal{M})_{\eta}$$

is the inclusion of the stratum for $\Lambda \subseteq \{1, \dots, r\}$.

Proof. Inductively apply the localization triangle

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*i^*\mathcal{F}$$

for an open/closed stratification (j being the open inclusion, i the closed inclusion). Repeating this produces $\mathcal{F}$ in terms of the thick generators. It is worth pointing out that in this procedure p -Kummer sheaves are closed under extensions, since invariants for $I_{\bar{x}}^{\log, (p)'}$ are exact (due to this being the prime-to- p part). $\square$

Thus far, in $D_{\text{fgu}}((\text{Spf } R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ we have obtained only one class of these generators for the open stratum. In the situation of a framed log affine, we will now introduce the categories needed to handle other strata.

Notation for this section. Consider a framed log affine $(\text{Spf } R, \mathcal{M})$ with framing $\square$ as in Definition 3.2, and put $A = R_0^{\square}((q-1))_p^{\wedge}$. For $\Lambda \subseteq \{1, \dots, r\}$, put

$$A_{\Lambda} = A / (T_i)_{i \in \Lambda} [1/T_j]_{p, j \notin \Lambda}^{\wedge}.$$

We introduce the category

$$D_{\text{fgu}}(A_{\Lambda})^{h\Gamma},$$

which actually by the Fitting ideal argument in Proposition 4.28 is simply $\text{Perf}(A_{\Lambda})^{\varphi=1, h\Gamma}$.

Ultimately we will argue these categories correspond to $j_{\Lambda,!} D_{\text{lis}}^{(b), p\text{Kum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p)$, up to a shift which only matters for perversity. Note that unlike the open stratum, we do actually need to pay attention to the p -Kummer condition on these more general strata. They are equipped with their pullback log structures, so there will be a hollow component to the log structure on which the p -Kummer condition is not vacuous. To properly embed these categories as full subcategories into $D_{\text{fgu}}(A)^{h\Gamma}$ we need a variant of Kashiwara's lemma.

We write $D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_Z$ for the full subcategory of finitely generated unit Laurent crystals supported on Z . Let us explain the general globalized notion of this when $D = \bigcup_i D_i$ is a simple normal crossings divisor with irreducible components D_i . For any nonempty intersection $Z = D_1 \cap \dots \cap D_c$ of distinct irreducible components, define

$$D_{\text{fgu}}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})_Z := \bigcap_{a=1}^c \ker \left(- \otimes^{\mathbf{L}, \square} \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}(*D_a) \right).$$

Similarly one may make a definition for connected components. This covers all Z which are smooth closed log strata of the stratification induced by D ; connected components are necessarily of pure codimension.

Under Theorem 4.7, we may translate the previous definition on a framed log affine to obtain the analogous category $D_{\text{fgu}}(A)_{J_{\Lambda}}^{h\Gamma}$ of finitely generated unit modules supported on T_i for $i \in \Lambda \subseteq \{1, \dots, r\}$, where $J_{\Lambda} = (T_i)_{i \in \Lambda}$. We then can prove the following:

Lemma 5.19 (Kashiwara's lemma). *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine, and as usual put $A = R_0^{\square}((q-1))_p^{\wedge}$ and fix $\Lambda \subset \{1, \dots, r\}$. We put $J_{\Lambda} = (T_i)_{i \in \Lambda}$.*

Then there is an equivalence

$$i^! := \left((A/J_{\Lambda}) \otimes_A^{\mathbf{L}} - \right) [-|\Lambda|] : D_{\text{fgu}}(A)_{J_{\Lambda}}^{h\Gamma} \xrightarrow{\sim} D_{\text{fgu}}(A/J_{\Lambda})^{h\Gamma}$$

Proof. Put $\bar{A} = A/p$ and $\bar{J}_{\Lambda} = J_{\Lambda} \bar{A}$. The categories and functor above satisfy the hypotheses of Lemma 2.3; the support condition is detected modulo p by derived Nakayama. It is therefore enough to prove that

$$i^! := \left((\bar{A}/\bar{J}_{\Lambda}) \otimes_{\bar{A}}^{\mathbf{L}, \square} - \right) [-|\Lambda|] : D_{\text{fgu}}(\bar{A})_{\bar{J}_{\Lambda}}^{h\Gamma} \longrightarrow D_{\text{fgu}}(\bar{A}/\bar{J}_{\Lambda})^{h\Gamma}$$

is an equivalence. The functor is well-defined: one can use Lemma 4.27, and in particular $i^!$ is Γ -equivariant.

In this case, the functor $i^!$ is an equivalence *before* we apply $(-)^{h\Gamma}$. One may use the proof of [EK04b, Corollary 5.11.3] to the present regular F-finite situation (noting we have access to important ingredients like roots by [Bli08]). Thus one obtains an adjoint equivalence

$$i_+ : D_{\text{fgu}}(\bar{A}/\bar{J}_\Lambda) \rightleftarrows D_{\text{fgu}}(\bar{A})_{\bar{J}_\Lambda} : i^!.$$

This survives after applying a cobar construction $(-)^{h\Gamma}$ to both sides, because evaluation on the degree zero term is conservative. Thus one may test $i^!i_+ \simeq \text{id}$ and $i_+i^! \simeq \text{id}$ after evaluation on the degree zero term, at which point it reduces to the ordinary Kashiwara lemma for finitely generated unit modules. $\square$

We use J_Λ as in Lemma 5.19. Applying the *inverse* i_+ in Kashiwara's lemma yields a more general fully faithful functor

$$D_{\text{fgu}}(A/J_\Lambda)^{h\Gamma, \text{op}} \xrightarrow{i_+} D_{\text{fgu}}(A)^{h\Gamma, \text{op}},$$

and there is an obvious inclusion, corresponding to $!$ -extension,

$$\text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} \hookrightarrow D_{\text{fgu}}(A/J_\Lambda)^{h\Gamma, \text{op}}.$$

We denote the composite functor, again fully faithful, by $j_{\Lambda, !}$.

The category $\text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}}$ is also equipped with a functor

$$T_{\text{ét}} : \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} \longrightarrow D_{\text{lisse}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p)$$

arising in the same way as Lemma 4.30.

Corollary 5.20. *The following diagram commutes:*

$$\begin{array}{ccc} \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} & \xrightarrow{j_{\Lambda, !}} & D_{\text{fgu}}(A)^{h\Gamma, \text{op}} \\ \downarrow T_{\text{ét}} & & \downarrow T_{\text{ét}} \\ D_{\text{lisse}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p) & \xrightarrow{j_{\Lambda, !}[-|\Lambda|]} & D_{\text{cons}, D}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\eta}, \mathbf{Z}_p) \end{array}$$

On the right $T_{\text{ét}}$ is passed through the equivalence of Theorem 4.7, which may then be explicitly computed via Lemma 4.30.

Proof. This is more or less clear by construction from Lemma 4.30, by expanding what the square means:

$$\begin{array}{ccccc} \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} & \xrightarrow{j_{\Lambda, !}} & D_{\text{fgu}}(A)^{h\Gamma, \text{op}} & & \\ \downarrow T_{\text{ét}}^{\text{perf}} & & \downarrow T_{\text{ét}}^{\text{perf}} & & \\ D_{\text{lisse}}^{(b)}(\text{Spec}(R_0^\square)_{\infty, \Lambda, \eta}^\flat, \mathbf{Z}_p)^{h\Gamma} & \xrightarrow{j_{\Lambda, !}[-|\Lambda|]} & D_{\text{cons}, D}^b(\text{Spec}(R_0^\square)_{\infty, \eta}^\flat, \mathbf{Z}_p)^{h\Gamma} & & \\ \downarrow \text{an}_\Gamma^* & & \downarrow \text{an}_\Gamma^* & & \\ D_{\text{lisse}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p) & \xrightarrow{j_{\Lambda, !}[-|\Lambda|]} & D_{\text{cons}, D}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\eta}, \mathbf{Z}_p) & \xleftarrow{T_{\text{ét}}} & \end{array}$$

Here $\mathrm{Spec}(R_0^\square)_{\infty, \Lambda, \eta}^b$ similarly means that we set take the reduced ring where $T_i^b = 0$ for $i \in \Lambda$, and invert them for $i \notin \Lambda$ (note this is not a perfectoid ring). By abuse of notation we base change to the perfection first and then apply $T_{\text{ét}}^{\text{perf}}$ – more precisely, in Lemma 4.30 we have denoted the composite of the first two arrows by simply $T_{\text{ét}}^{\text{perf}}$.

The bottom square is commutative, so we reduce to checking commutativity of the top square. This is clear for the first component of

$$j_{\Lambda,!} : \mathrm{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} \hookrightarrow D_{\text{fgu}}(A/J_\Lambda)^{h\Gamma, \text{op}} \xrightarrow{i_+} D_{\text{fgu}}(A)^{h\Gamma, \text{op}}$$

so we reduce to studying compatibility for i_+ .

One can further reduce to checking this modulo p by derived Nakayama, as there is already a morphism we can test is an equivalence to witness commutativity. Indeed for the closed part i_+ of our construction of $j_{\Lambda,!}$ there is a map

$$T_{\text{ét}}^{\text{perf}}(i_+M) \longrightarrow i_* i^* T_{\text{ét}}^{\text{perf}}(i_+M) \longrightarrow i_* T_{\text{ét}}^{\text{perf}}(A/J_\Lambda \otimes_A^L i_+M) = i_* T_{\text{ét}}^{\text{perf}}(M)[-|\Lambda|].$$

This is then not too difficult, because one can see that for the deperfedted Emerton-Kisin solution functor of [EK04b] one has for $M \in \mathrm{Perf}(\overline{A}_\Lambda)^{h\Gamma, \varphi=1}$ that

$$\mathrm{RSol}(i_+M) = i_* \mathrm{RSol}(M)[-|\Lambda|],$$

compatibly with the Γ -action. On the left we use the solution functor for $D_{\text{fgu}}(\overline{A})$ and on the right for $D_{\text{fgu}}(\overline{A}/J_\Lambda)^{h\Gamma}$. By adjunction one can reduce the check to one for $i^!$ (revealing the source of the shift).

For the comparison we actually needed to work with $T_{\text{ét}}^{\text{perf}}$, but base changing to $\mathrm{Spec}(R_0^\square)_{\infty, \eta}^b$ preserves this identity. $\square$

In fact, the left vertical arrow in the diagram of Corollary 5.20 is an equivalence. This can be proved directly.

Proposition 5.21. *The functor*

$$T_{\text{ét}} : \mathrm{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} \longrightarrow D_{\text{lis}}^{(b), \text{pKum}}((\mathrm{Spf} R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p)$$

is an equivalence.

Proof. We first check the case $|\Lambda| = r$, which we will explain in full detail. Here this is because the corresponding stratum on the log adic generic fiber is an ordinary framed affine $Z = \bigcap_{i=1}^r D_i$ over a rank $n - r$ torus with a rank r hollow log structure. Equipping the target torus with the hollow chart $\mathbf{N}^r \rightarrow 0$, we get an induced strict étale framing

$$\square : (Z, \mathcal{M}|_Z) \longrightarrow (\mathrm{Spf} W(k)[\zeta_p] \langle T_{r+1}^\pm, \dots, T_n^\pm \rangle, (\mathbf{N}^r \rightarrow 0)^a).$$

The torus has a natural perfectoid cover $\mathrm{Spf} W(k)^{\text{cyc}} \langle T_{r+1}^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty} \rangle$ with hollow log structure $\mathbf{N}[1/p]^r \rightarrow 0$, so using $\square$ we obtain an associated perfectoid cover $(R_0^\square)_{\infty, Z}[1/p]$ again with the chart $\mathbf{N}[1/p]^r \rightarrow 0$; we let $(A/J_\Lambda)_{\text{perf}}$ denote $\mathbb{A}_{\text{inf}}((R_0^\square)_{\infty, Z}[1/p])$. Katz's theorem, tilting, and pro-Kummer-étale descent for the same full group Γ give

$$(\mathrm{Perf}((A/J_\Lambda)_{\text{perf}})^{\varphi=1, h\Gamma})^{\text{op}} \xrightarrow{\sim} D_{\text{lis}}^{(b), \text{pKum}}((Z, \mathbf{N}^r)_\eta, \mathbf{Z}_p) \xrightarrow{i_*} D_{\text{cons}, D}^{(b)}((\mathrm{Spf} R, \mathcal{M})_\eta, \mathbf{Z}_p).$$

In this case the finitely generated unit condition is equivalent to being perfect with unit Frobenius by the proof of Proposition 4.28. Moreover,

$$D_{\text{fgu}}(A/J_\Lambda)^{h\Gamma} \simeq \text{Perf}((A/J_\Lambda)_{\text{perf}})^{\varphi=1, h\Gamma}$$

by [BS23, Proposition 3.6] and Lemma 2.3; thus the desired claim follows. We note Lemma 4.30 identifies this construction with $T_{\text{ét}}$.

In the remaining cases where $|\Lambda|$ is smaller we must involve part of the open stratum theorem, but fortunately the hard work has already been done. The result we need is that

$$\text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} \xrightarrow{\sim} D_{\text{lis}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p),$$

where the target has its natural hollow pullback log structure. The equivalence on the open stratum tells us that there is an equivalence

$$\text{Perf}(A_\Lambda)^{h\Gamma', \varphi=1, \text{op}} \xrightarrow{\sim} D_{\text{lis}}^{(b)}((\text{Spf } R)_{\Lambda, \eta}, \mathbf{Z}_p)$$

where on the right we use the trivial log structure and Γ' is the quotient $(1 + p\mathbf{Z}_p) \ltimes \mathbf{Z}_p(1)^{n-|\Lambda|}$. We can check the equivalence modulo p by Lemma 2.3. Full faithfulness can be done through exactly the same argument as Theorem 5.4. The only nontrivial point is to note that adding hollow directions to Proposition 5.9 does not affect the reduction argument. Moreover in the helper lemmas, since we work modulo p by increasing the parameter m we can make the hollow part act trivially after shrinking it. This follows because the action of $\mathbf{Z}_p(1)^{|\Lambda|}$ factors through a finite p -group modulo p . The hollow directions will then not affect the remainder of the argument, which can be carried out in the same way.

With full faithfulness in hand, essential surjectivity is easier. Objects in $D_{\text{lis}}^{(b)}((\text{Spf } R)_{\Lambda, \eta}, \mathbf{F}_p)$ can be regarded as ones in $D_{\text{lis}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{F}_p)$ where hollow log inertia acts trivially. We will roughly argue that these thickly generate to deduce essential surjectivity.

The key point is again that $\mathbf{Z}_p(1)^{|\Lambda|}$ acts on a finite dimensional $\mathbf{F}_p$ -vector space V through a finite p -group H , giving a finite filtration where the action on graded pieces is trivial. To construct this filtration we repeatedly use that fact that because H is a finite p -group we always have $V^H \neq 0$ for nonzero V ; this can be seen by applying the orbit-stabilizer theorem. We can filter an H -representation V as follows: start with $0 \subset \text{Fil}^1 V = V^H$, and then use the exact sequence

$$0 \longrightarrow V^H \longrightarrow \text{Fil}^2 V \longrightarrow (V/V^H)^H \longrightarrow 0$$

of H -representations, where $\text{Fil}^2 V \subseteq V$ is the preimage of $(V/V^H)^H$ under $\pi : V \rightarrow V/V^H$. Using the same fact, $(V/V^H)^H \neq 0$. Continuing in this manner, since $\dim_{\mathbf{F}_p} V$ is finite we get a finite H -equivariant filtration $\text{Fil}^\bullet V$ where graded pieces have a trivial action by H . In our situation V is actually a representation of the larger group in $\pi_1^{\text{két}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \bar{x})$, but since $\mathbf{Z}_p(1)^{|\Lambda|}$ is normal in $\pi_1^{\text{két}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, x)$ via

$$1 \longrightarrow \widehat{\mathbf{Z}}(1)^{|\Lambda|} \longrightarrow \pi_1^{\text{két}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}((\text{Spf } R)_{\Lambda, \eta}, \bar{x}) \longrightarrow 1$$

this argument yields a finite filtration $\text{Fil}^\bullet V$ for p -Kummer $V \in \text{Rep}_{\mathbf{F}_p}^{\text{fin}}(\pi_1^{\text{két}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \bar{x}))$ with associated graded pieces in $\text{Rep}_{\mathbf{F}_p}^{\text{fin}}(\pi_1^{\text{ét}}((\text{Spf } R)_{\Lambda, \eta}, \bar{x}))$.¹⁰ Using derived full faithfulness, this filtration suffices to show all such p -Kummer V are in the essential image. By a Postnikov dévissage we reduce to essential surjectivity in the heart, so the claim follows. $\square$

¹⁰Recall p -Kummer means that the prime-to- p part $\widehat{\mathbf{Z}}^{(p)'}(1)^{|\Lambda|}$ already acts trivially.

Remark 5.22. We can also argue essential surjectivity just as in the proof of Theorem 5.4 using $\mu_N^{r-|\Lambda|}$ descent after taking a prime-to- p Kummer cover. To make this fully rigorous requires dealing with more technical details than above, however, which is why we have not used this argument.

Combining Proposition 5.21 and Corollary 5.20 we will have deduced essential surjectivity, if we can prove full faithfulness. We tackle this next.

In Lemma 5.17, we must now consider the different types of generators we need to compare. We also have not yet shown that $\text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1}$, properly embedded via $j_{\Lambda,!}$, thickly generate $D_{\text{fgu}}(A)^{h\Gamma}$. We establish this first; the proof method is actually nearly identical to Lemma 5.18.

For a closed subset $i : Z_\Lambda \hookrightarrow \text{Spf } R$ cut out by $J_\Lambda = (T_i)_{i \in \Lambda}$ and $M \in D_{\text{fgu}}(A/J_\Lambda)^{h\Gamma}$, the equivalence above in Lemma 5.19 gives

$$T_{\text{ét}}(i_+ M) \simeq i_* T_{\text{ét}}(M)[-|\Lambda|].$$

It is also easy to give a formula for $i_+ i^!(-)$ even when the input is not supported on Z . On the same chart, local cohomology along Z is computed by

$$R\Gamma_Z(M) \simeq M \otimes_A^L \bigotimes_{i \in \Lambda}^L [A \rightarrow (A[1/T_i])_p^\wedge].$$

Completed base change to A/J_Λ kills every localized term, so $i^! R\Gamma_Z(M) \simeq i^! M$. The complex $R\Gamma_Z(M)$ is canonically Γ -equivariant, since each $\gamma \in \Gamma$ carries every T_i to a unit multiple of itself. Checking $R\Gamma_Z(M)$ is finitely generated unit is straightforward, so Kashiwara's lemma therefore gives

$$R\Gamma_Z(M) \simeq i_+ i^! M.$$

Lemma 5.23. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine. The category $D_{\text{fgu}}(A)^{h\Gamma}$ is thickly generated by $j_{\Lambda,!} M_\Lambda$ for $M_\Lambda \in \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1}$.*

Proof. Similar to Lemma 5.18, we need to produce analogous triangles extracting open and closed parts. Let $M \in D_{\text{fgu}}(A)^{h\Gamma}$.

For an irreducible component D_i in D , there is an exact triangle

$$R\Gamma_{D_i}(M) \rightarrow M \rightarrow M(*D_i),$$

extracting $M(*D_i)$ supported away from D_i and $R\Gamma_{D_i}(M)$ supported on D_i (we implicitly use Lemma 5.19 here to see that this part comes from $D_{\text{fgu}}(A/T_i)^{h\Gamma}$). Repeating this for collections $\{D_i\}_{i \in \Lambda}$ we can eventually show that M is in the thick subcategory generated by $j_{\Lambda,!} M_\Lambda$ for $M_\Lambda \in \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1}$.

Once we reach an individual stratum, a priori we only have $M_\Lambda \in D_{\text{fgu}}(A_\Lambda)^{h\Gamma}$. Fortunately the Fitting ideal argument in Proposition 4.28 shows that these are necessarily perfect. Then we run the same argument: the Γ -invariant ideals are either zero or the unit ideal on connected components of the scheme. For an object $M \in \text{Mod}_{\text{fgu}}(\overline{A}_\Lambda)^{h\Gamma}$ admits a Γ -equivariant minimal root $\beta : M_0 \rightarrow \varphi^* M_0$. Taking Fitting ideals, we see the source and target are vector bundles of the same rank and coker β is a vector bundle. Using ranks the root map is an equivalence, so all of the objects in the heart are vector bundles modulo p . By a Postnikov dévissage, this proves

perfectness modulo p . We can then test perfectness modulo p since A_Λ is still derived p -complete (as well as M_Λ), so the desired result follows. $\square$

We have now established everything in Lemma 5.17 that we need to establish the main theorem by comparing thick generators, except the comparisons of RHom s on generators. For these, the general strategy will be a reduction to one of the known equivalences in Proposition 5.21. For this reduction, we will need RHom comparisons on both sides of the correspondence. We work out the Kummer-étale side first, using the new notation

$$Z_{\Lambda,\eta} = \bigcap_{\lambda \in \Lambda} D_{\lambda,\eta} \subset (\mathrm{Spf} R, \mathcal{M})_\eta.$$

Lemma 5.24. *Consider a framed log affine $(\mathrm{Spf} R, \mathcal{M})$, and choose (possibly empty) subsets $\Lambda \subseteq \Lambda' \subseteq \{1, \dots, r\}$. Letting $\mathbb{L}' \in D_{\mathrm{lis}}^{(b)}((\mathrm{Spf} R, \mathcal{M})_{\Lambda',\eta})$ and $\mathbb{L}$ be derived lisse on $Z_{\Lambda,\eta} \subset (\mathrm{Spf} R, \mathcal{M})_\eta$, we have*

$$\mathrm{RHom}(j_{\Lambda',!}\mathbb{L}', j_{\Lambda,!}j_{\Lambda}^*\mathbb{L}) \simeq \mathrm{RHom}(j_{\Lambda',!}\mathbb{L}', i_{Z_{\Lambda',\eta}}^*\mathbb{L})[-|\Lambda' \setminus \Lambda|].$$

Here $j_\Lambda, j_{\Lambda'}$ are the inclusions of the strata $(\mathrm{Spf} R, \mathcal{M})_{\Lambda,\eta}$ and $(\mathrm{Spf} R, \mathcal{M})_{\Lambda',\eta}$ with their pullback log structures.

Proof. We can reframe this by working in $Z_{\Lambda,\eta} \setminus \bigcup_{j \notin \Lambda'} D_{j,\eta}$. The point of doing this is so that $(\mathrm{Spf} R, \mathcal{M})_{\Lambda',\eta}$ has its inclusion $i_{\Lambda'}$ become a strict closed immersion, and $(\mathrm{Spf} R, \mathcal{M})_{\Lambda,\eta}$ has its inclusion j_Λ become a strict open immersion. This makes their roles more transparent in what follows.

One checks

$$j_{\Lambda,!}j_{\Lambda}^*\mathbb{L} \simeq \left[\mathbb{L} \longrightarrow \bigoplus_i \mathbb{L}|_{D_{i,\eta}} \longrightarrow \bigoplus_{i < j} \mathbb{L}|_{D_{i,\eta} \cap D_{j,\eta}} \longrightarrow \dots \longrightarrow i_{\Lambda',*}i_{\Lambda'}^*\mathbb{L} \right].$$

Here the direct sums are indexed over members of $\Lambda' \setminus \Lambda$. This can be seen for example by checking stalks on both sides. A similar resolution appears in [Yam11, §2].

If one substitutes this resolution in $\mathrm{RHom}(i_{\Lambda',*}\mathbb{L}', j_{\Lambda,!}j_{\Lambda}^*\mathbb{L})$, everything except the last term vanishes (which gives the shift). Vanishing can be checked modulo p and after a Postnikov dévissage to the heart, where we use logarithmic purity as in [DLLZ23, Corollary 4.6.7]. The present hollow case can again be deduced from this by keeping track of hollow log inertia. For example to deduce the claim for $Z_{\Lambda,\eta}^\circ \xrightarrow{j_\Lambda} Z_{\Lambda,\eta}$ we instead equip them both with the log structure induced by $\bigcup_{i \notin \Lambda} D_{i,\eta}$ (which has no hollow component), so the log structure on $Z_{\Lambda,\eta}^\circ$ becomes trivial. Then [DLLZ23, Corollary 4.6.7] applies, and on categories of derived lisse pro-Kummer-étale sheaves we take $\mathrm{Mod}_{\mathbb{Z}_p[\widehat{\mathbb{Z}}(1)^{|\Lambda|}]}(-)$; this proves the purity isomorphism $\mathbb{L} \simeq \mathrm{R}j_{\Lambda,*}j_{\Lambda}^*\mathbb{L}$ we want. When we intersect k additional $D_{i,\eta}$ we take $\widehat{\mathbb{Z}}(1)^{|\Lambda|+k}$. $\square$

This reduces to an RHom between local systems on the stratum corresponding to Λ' , because now the inclusion of $(X, \mathcal{M})_{\Lambda',\eta} \hookrightarrow Z_{\Lambda',\eta}$ is a strict open immersion. It is worth pointing out that the case handled above where $\Lambda \subseteq \Lambda'$ is the only interesting case, as in other cases the RHom is zero.

Recall we have already fully faithfully embedded $\mathrm{Perf}(A_\Lambda)^{h\Gamma, \varphi=1}$ for all Λ into $D_{\mathrm{fgu}}(A)^{h\Gamma}$ using the construction preceding Corollary 5.20. The next lemma repeats Lemma 5.24 but on the finitely generated unit module side.

Lemma 5.25. *Consider a framed log affine $(\mathrm{Spf} R, \mathcal{M})$, and choose (possibly empty) subsets $\Lambda \subseteq \Lambda' \subseteq \{1, \dots, r\}$. Let $M_\Lambda \in \mathrm{Perf}(A/(T_i)_{i \in \Lambda})^{h\Gamma, \varphi=1}$, and let $M_{\Lambda'} \in \mathrm{Perf}(A_{\Lambda'})^{h\Gamma, \varphi=1}$, and regard both as objects in $D_{\mathrm{fgu}}(A/(T_i)_{i \in \Lambda})^{h\Gamma}$ using Kashiwara's lemma (Lemma 5.19). Then*

$$\mathrm{RHom}(M_\Lambda(*D_\Lambda), M_{\Lambda'}) \simeq \mathrm{RHom}(\mathrm{R}\Gamma_{\cap_{j \in \Lambda' \setminus \Lambda} D_j} M_\Lambda, M_{\Lambda'})[-|\Lambda' \setminus \Lambda|].$$

Proof. The argument is more or less identical, but translated to the finitely generated unit side. We no longer have access to logarithmic purity so we will need to prove the required vanishing in a different way.

For convenience of notation put $|\Lambda' \setminus \Lambda| = c$. Without affecting any of the results, we may instead work over $A_{\Lambda, \Lambda'} := A/(T_i)_{i \in \Lambda}[1/T_j]_{p, j \notin \Lambda'}^\wedge$, since $M_{\Lambda'}$ is already localized at these variables. This is the analogue in the previous proof of passing to $Z_{\Lambda, \eta} \setminus \bigcup_{j \notin \Lambda'} D_{j, \eta}$.

By tensoring with the $(\Gamma$ -equivariant) localization triangles

$$\mathrm{R}\Gamma_{D_i}(A_{\Lambda, \Lambda'}) \longrightarrow A_{\Lambda, \Lambda'} \longrightarrow A_{\Lambda, \Lambda'}(*D_i)$$

for $i \in \Lambda' \setminus \Lambda$ we see that

$$M_\Lambda(*\bigcup_{i \in \Lambda' \setminus \Lambda} D_i) \simeq \left[\mathrm{R}\Gamma_{\cap_{i \in \Lambda' \setminus \Lambda} D_i} M_\Lambda \longrightarrow \bigoplus_{|I|=c-1} \mathrm{R}\Gamma_{\cap_{i \in I} D_i} M_\Lambda \longrightarrow \dots \longrightarrow \bigoplus_i \mathrm{R}\Gamma_{D_i} M_\Lambda \longrightarrow M_\Lambda \right]$$

where we always index over $I \subseteq \Lambda' \setminus \Lambda$, and the resolution is in degrees $-c$ to zero. For the purposes of mapping out to $M_{\Lambda'}$, this can be used in place of $M_\Lambda(*D_\Lambda)$.

What remains is an orthogonality result: the desired result follows if we show only the first term has nonzero $\mathrm{RHom}(-, M_{\Lambda'})$.

Abstractly, for $\mathrm{R}\Gamma_{\cap_{i \in I} D_i} M_\Lambda$ one sees by Kashiwara that

$$\mathrm{RHom}(\mathrm{R}\Gamma_{\cap_{i \in I} D_i} M_\Lambda, M_{\Lambda'})$$

can be computed as an RHom in $A_{\Lambda, \Lambda'}/(T_i)_{i \in I}$ where the first argument is perfect and the second argument is supported on D_j for $j \notin I$ (here we assume I is proper). This follows from Lemma 5.26, after reducing to working modulo p . $\square$

Lemma 5.26. *Let $\overline{A}$ be a regular F -finite $k((q-1))$ algebra, with a continuous action of $\Gamma = (1+p\mathbb{Z}_p) \ltimes \mathbb{Z}_p(1)^n$. Let $T \in \overline{A}$ be a nonzerodivisor such that all generators of Γ act trivially, except for γ_i generating the i th copy of $\mathbb{Z}_p(1)_i$ which acts by*

$$\gamma_i \cdot T = q^{p^i} T$$

and fixes $\overline{A}/T$. Let $M \in \mathrm{Perf}(\overline{A})^{\varphi=1, h\Gamma}$ and $M_i \in D_{\mathrm{fgu}}(\overline{A}/T)^{h\Gamma}$. Then

$$\mathrm{RHom}(M, i_+ M_i) = 0.$$

We note the proof of Kashiwara's lemma still provides i_+ under these hypotheses, as we used nothing more than these hypotheses.

Proof. Using Lemma 4.2 and the Emerton-Kisin correspondence in the regular case, established in [Mat23, Theorem 6.20], before taking Γ -invariants we identify the mapping complex $\mathrm{RHom}(\mathrm{M}, i_+ \mathrm{M}_i)$ of finitely generated unit Frobenius modules over $\bar{A}$ with $\mathrm{RHom}(i_* \mathcal{F}, \mathrm{L})$ on $X = \mathrm{Spec} \bar{A}$. Here $\mathcal{F}$ is a bounded constructible complex of étale $\mathbf{F}_p$ -sheaves on $Z = \mathrm{Spec}(\bar{A}/(T))$ and L is a bounded complex with lisse cohomology on $X = \mathrm{Spec} \bar{A}$; we absorb the shift in Kashiwara's lemma into $\mathcal{F}$.

There is a subtlety in translating continuity to the étale side, but we can avoid it here: it is enough to show that $\gamma_i - 1$ acts invertibly on $\mathrm{RHom}(i_* \mathcal{F}, \mathrm{L})$, where γ_i is the chosen topological generator of $\mathbf{Z}_p(1)_i$. Naturality transfers this operator through the Emerton-Kisin correspondence. The Koszul resolution (1) for the toric subgroup $\mathbf{Z}_p(1)^n$, followed by Hochschild-Serre, then gives the desired vanishing.

By adjunction and the fact that L is derived lisse, we have

$$\mathrm{RHom}_X(i_* \mathcal{F}, \mathrm{L}) \simeq i_* \mathrm{RHom}_Z(\mathcal{F}, i^! \mathrm{L}) \simeq i_* \mathrm{RHom}_Z(\mathcal{F}, i^* \mathrm{L} \otimes_{\mathbf{F}_p}^{\mathrm{L}} i^! \mathbf{F}_p).$$

A Postnikov dévissage in $\mathcal{F}$ and L reduces the claim to $\mathcal{F}$ constructible and L lisse, both concentrated in degree zero. The group $\mathbf{Z}_p(1)_i$ fixes Z , so its actions on $\mathcal{F}$ and $i^* \mathrm{L}$ factor through finite automorphism groups: a constructible $\mathbf{F}_p$ -sheaf on a noetherian scheme has finite dimensional $\mathbf{F}_p$ -vector space of endomorphisms, as one sees from a finite stratification and one geometric stalk on each connected stratum. Thus some $H = p^\ell \mathbf{Z}_p(1)_i$ acts trivially on both. Put $\gamma = \gamma_i^{p^\ell}$. It remains to show that $\gamma - 1$ acts invertibly on $i^! \mathbf{F}_p$.

For a geometric point $\bar{z}$ of Z , put $B = \mathcal{O}_{X, \bar{z}}^{\mathrm{sh}}$ and $K = (B[1/T]/B)[-1]$. Using the triangle

$$i_* i^! \mathbf{F}_p \longrightarrow \mathbf{F}_p \longrightarrow \mathrm{R}j_* \mathbf{F}_p$$

upon taking the stalk at $\bar{z}$ and using Artin-Schreier to compute $(\mathbf{F}_p)_{\bar{z}}$ and $(\mathrm{R}j_* \mathbf{F}_p)_{\bar{z}}$ as mapping fibers $\mathrm{fib}(\varphi - 1 : B \longrightarrow B)$ and $\mathrm{fib}(\varphi - 1 : B[1/T] \longrightarrow B[1/T])$ respectively, the fact that fibers commute with fibers yields

$$(i^! \mathbf{F}_p)_{\bar{z}} \simeq \mathrm{fib}(\varphi - 1 : K \longrightarrow K).$$

Since γ fixes $B/(T)$ and sends T to $q^{p^{\ell+1}} T$, the operator $\gamma - 1$ acts by the units $q^{-mp^{\ell+1}} - 1$ on the graded pieces of the filtration $T^{-m} B/B$ on $B[1/T]/B$. It is therefore invertible on K and, as it commutes with Frobenius, on $(i^! \mathbf{F}_p)_{\bar{z}}$. This proves the claim on stalks.

Thus $\gamma - 1$ acts invertibly on $\mathrm{RHom}_X(i_* \mathcal{F}, \mathrm{L})$. Finally, $\gamma - 1 = (\gamma_i - 1)^{p^\ell}$ in characteristic p , so $\gamma_i - 1$ is also invertible. Taking global sections and applying the preceding group cohomology calculation proves the lemma. $\square$

With Lemma 5.24 and Lemma 5.25 in hand, we can now prove the required RHom comparisons by making a reduction to the known comparison on $\mathrm{Perf}(A_{\Lambda'})^{\varphi=1, h\Gamma}$ (with Λ' as in the notation of the theorem below).

Theorem 5.27. *Choose (possibly empty) subsets $\Lambda, \Lambda' \subseteq \{1, \dots, r\}$. Let $M_\Lambda \in \mathrm{Perf}(A_\Lambda)^{h\Gamma, \varphi=1}$, and let $M_{\Lambda'} \in \mathrm{Perf}(A_{\Lambda'})^{h\Gamma, \varphi=1}$. Then $T_{\mathrm{ét}}$ induces an equivalence*

$$\mathrm{RHom}_{\mathrm{D}_{\mathrm{fgu}}(A)^{h\Gamma}}(j_{\Lambda, !} M_\Lambda, j_{\Lambda', !} M_{\Lambda'}) \simeq \mathrm{RHom}(T_{\mathrm{ét}}(j_{\Lambda', !} M_{\Lambda'}), T_{\mathrm{ét}}(j_{\Lambda, !} M_\Lambda)).$$

Proof. The comparisons we need fit into three cases: where $\Lambda = \Lambda'$, where $\Lambda \subseteq \Lambda'$, and where neither holds or $\Lambda \not\subseteq \Lambda'$. In the third case, on both sides it is easy to see that the RHom is zero. In the first case we have already proven this by combining Corollary 5.20 and Proposition 5.21.

Thus, all of the remaining work is the case $\Lambda \subseteq \Lambda'$. Geometrically, on the Kummer-étale side both $T_{\text{ét}}(j_{\Lambda'}!M_{\Lambda'})$ and $T_{\text{ét}}(j_{\Lambda}!M_{\Lambda})$ are supported on the closed locus

$$Z_{\Lambda,\eta} = \bigcap_{\lambda \in \Lambda} D_{\lambda,\eta} \subset (\mathrm{Spf} R, \mathcal{M})_{\eta}$$

equipped with the pullback log structure. Inside of this closed locus on the Kummer-étale side of the correspondence, the RHom we are computing involves maps from $j_{\Lambda'}!\mathbb{L}' \rightarrow j_{!,\Lambda}\mathbb{L}$ (up to some shifts). In other words, maps from an arbitrary deeper stratum to $\mathbb{L}$ supported on the open stratum $(X, \mathcal{M})_{\Lambda,\eta} = Z_{\Lambda,\eta}^{\circ} \subset Z_{\Lambda,\eta}$. Via Kashiwara's lemma, we may therefore drop $j_{\Lambda'}!$ and $j_{!,\Lambda}$ and simply consider objects $M_{\Lambda'}, M_{\Lambda}$ in $D_{\mathrm{fgu}}(A/(T_i)_{i \in \Lambda})^{h\Gamma}$ supported on the deeper stratum $(X, \mathcal{M})_{\Lambda',\eta} \subset Z_{\Lambda,\eta}$ and the open stratum respectively.

We first argue in a special case where M_{Λ} admits an extension to

$$M_{\Lambda}^{\mathrm{ext}} \in \mathrm{Perf}(A/(T_i)_{i \in \Lambda})^{\varphi=1, h\Gamma}.$$

This means M_{Λ} is recovered as $M_{\Lambda}^{\mathrm{ext}} \otimes A_{\Lambda} = M_{\Lambda}^{\mathrm{ext}}(*D_{\Lambda})$, i.e. we p -completely invert the remaining coordinates. Geometrically, this asks on the Kummer-étale side that it extends to a derived lisse sheaf on $Z_{\Lambda,\eta}$ with its pullback log structure.

In this situation, in $D_{\mathrm{fgu}}(A/(T_i)_{i \in \Lambda})^{h\Gamma}$ we have the equivalence

$$\mathrm{RHom}(M_{\Lambda}^{\mathrm{ext}}(*D_{\Lambda}), M_{\Lambda'}) \simeq \mathrm{RHom}(R\Gamma_{\bigcap_{j \in \Lambda' \setminus \Lambda} D_j} M_{\Lambda}^{\mathrm{ext}}, M_{\Lambda'})[-|\Lambda' \setminus \Lambda|].$$

provided by Lemma 5.25. On the right, using Kashiwara again gives an RHom in $D_{\mathrm{fgu}}(A/(T_i)_{i \in \Lambda'})^{h\Gamma}$, corresponding to a smaller closed locus. But then the target is in

$$\mathrm{Perf}(A_{\Lambda'})^{\varphi=1, h\Gamma} \subset D_{\mathrm{fgu}}(A/(T_i)_{i \in \Lambda'})^{h\Gamma},$$

and in particular using the localization adjunction we obtain the RHom

$$\mathrm{RHom}(R\Gamma_{\bigcap_{j \in \Lambda' \setminus \Lambda} D_j} M_{\Lambda}^{\mathrm{ext}}(*D_{\Lambda'}), M_{\Lambda'})[-|\Lambda' \setminus \Lambda|],$$

now taken in $\mathrm{Perf}(A_{\Lambda'})^{\varphi=1, h\Gamma}$ (so both arguments are being implicitly passed through Lemma 5.19).

For this final RHom we know $T_{\text{ét}}$ is fully faithful. The reduction will also work on the Kummer-étale side as well, which follows from Lemma 5.24. In particular, we obtain the analogous statement

$$\mathrm{RHom}(j_{\Lambda'}!\mathbb{L}', j_{\Lambda}!j_{\Lambda}^*\mathbb{L}^{\mathrm{ext}}) \simeq \mathrm{RHom}(j_{\Lambda'}!\mathbb{L}', i_{Z_{\Lambda',\eta}}^*\mathbb{L}^{\mathrm{ext}})[-|\Lambda' \setminus \Lambda|],$$

where the target RHom occurs on the closed locus $Z_{\Lambda',\eta}$, and the source on $Z_{\Lambda,\eta}$. Further restricting to the open locus $(X, \mathcal{M})_{\Lambda',\eta} \subset Z_{\Lambda',\eta}$, due to the supports this simplifies to a shifted RHom of derived lisse sheaves on $(X, \mathcal{M})_{\Lambda',\eta}$. These two final RHom s we obtained, on either side, can then be canonically identified using Proposition 5.21. Indeed they now involve derived lisse sheaves or objects in $\mathrm{Perf}(A_{\Lambda'})^{\varphi=1, h\Gamma}$, and using Corollary 5.20 they can be seen to be identified by $T_{\text{ét}}$.

The general case then follows from Lemma 5.28, which reduces the claim to the situation where $M_{\Lambda}^{\mathrm{ext}}$ exists. Indeed on the Kummer-étale side the same formula for μ_N^r invariants holds, precisely because

$$(\mathrm{Spf} R_N, \mathcal{M}_N)_{\eta} \longrightarrow (\mathrm{Spf} R, \mathcal{M})_{\eta}$$

is a torsor in the pro-Kummer-étale topology for this group. $\square$

We owe an additional helper lemma, which let us assume that M_Λ admits an extension. We let A_N be the ring of the (Laurentized) log q -de Rham prism of the Kummer cover as in Proposition 5.15. We also use the same notation for the Kummer cover.

Lemma 5.28. *Choose subsets $\Lambda \subseteq \Lambda' \subseteq \{1, \dots, r\}$, and suppose we are given objects $M_{\Lambda'}$ and M_Λ in $D_{\text{fgu}}(A/(T_i)_{i \in \Lambda})^{h\Gamma}$ supported on the strata for Λ' and Λ respectively. There is a prime-to- p Kummer cover $f : (\text{Spf } R_N, \mathcal{M}_N) \rightarrow (\text{Spf } R, \mathcal{M})$ such that*

$$f^* M_\Lambda = M_{\Lambda, N}^{\text{ext}}(*D_{\Lambda, N})$$

where $M_{\Lambda, N}^{\text{ext}} \in \text{Perf}((A_N/(T_i^{1/N})_{i \in \Lambda})^{h\Gamma, \varphi=1})$. Moreover,

$$\text{RHom}(f^* M_\Lambda, f^* M_{\Lambda'})^{h\mu_N^r} = \text{RHom}(M_\Lambda, M_{\Lambda'}).$$

Proof. The first assertion is straightforward. Write

$$Z_{\Lambda, \eta} = \bigcap_{\lambda \in \Lambda} D_{\lambda, \eta}$$

equipped with its pullback log structure as before.

Using Corollary 5.20, since in the diagram

$$\begin{array}{ccc} \text{Perf}((A/(T_i)_{i \in \Lambda})^{h\Gamma, \varphi=1, \text{op}} & \xrightarrow{T_{\text{ét}}} & D_{\text{lis}}^{(b), \text{pKum}}(Z_{\Lambda, \eta}, \mathbf{Z}_p) \\ \downarrow & & \downarrow j_\Lambda^* \\ \text{Perf}(A_\Lambda)^{h\Gamma, \varphi=1, \text{op}} & \xrightarrow{T_{\text{ét}}} & D_{\text{lis}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p) \end{array}$$

both horizontal arrows are equivalences we must check that $\mathbb{L} \in D_{\text{lis}}^{(b), \text{pKum}}((\text{Spf } R, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p)$ admits an extension $\mathbb{L}^{\text{ext}}$ which is again p -Kummer. Strictly speaking we did not show the top arrow is an equivalence, but because we have not inverted any T_i the proof is the same argument as the $|\Lambda| = r$ case of Proposition 5.21, but with the log perfectoid cover of $Z_{\Lambda, \eta}$ being the zero locus of T_i^{1/p^∞} for $i \in \Lambda$ in $(X, \mathcal{M})_{\infty, \eta}$ with its pullback log structure.

As we did in Proposition 5.21, keeping track of log inertia we reduce to the general assertion that for a framed log affine $(\text{Spf } R, \mathcal{M})$ and $\mathbb{L} \in D_{\text{lis}}^{(b)}(X_\eta \setminus D_\eta, \mathbf{Z}_p)$ there is a prime-to- p Kummer cover $f : (\text{Spf } R_N, \mathcal{M}_N) \rightarrow (\text{Spf } R, \mathcal{M})$ so $f^* \mathbb{L}$ admits a p -Kummer extension to $(\text{Spf } R_N, \mathcal{M}_N)_\eta$. By [DLLZ23, Corollary 6.3.4], the cohomology sheaves of $\mathbb{L}$ extend across the boundary as Kummer-étale local systems. The vanishing in [DLLZ23, Theorem 4.6.1], applied modulo p^n and passed to the inverse limit, makes the spectral sequence

$$E_2^{a, b} = R^a j_* H^b(\mathbb{L}) \longrightarrow H^{a+b}(Rj_* \mathbb{L})$$

degenerate. Thus $Rj_* \mathbb{L}$ is derived lisse, and, as in the essential surjectivity component of Theorem 5.4, for every log geometric point $\bar{x}$ and $b \in \mathbf{Z}$,

$$\text{im} \left(I_{\bar{x}}^{\log, (p)'} \longrightarrow \text{Aut}_{\mathbf{Z}_p} (H^b(Rj_* \mathbb{L})_{\bar{x}}) \right)$$

is finite. It follows that for each cohomology sheaf there is some prime-to- p Kummer cover making it p -Kummer by [IKY26, Proposition 5]. Appealing to a Postnikov dévissage as in [Hau24, Lemma 5.4.3] and boundedness, a single prime-to- p Kummer cover $\mathrm{Spf} R_N$ makes $R_{j*}\mathbb{L}$ p -Kummer. Thus, the first claim follows.

Similarly to Proposition 5.15, we have

$$A_N = \bigoplus_{\alpha \in (\mathbb{Z}/N\mathbb{Z})^r} A \cdot S^\alpha,$$

with a natural μ_N^r action associated to the grading, which gives $\mathrm{RHom}(f^*M_\Lambda, f^*M_{\Lambda'})$ a μ_N^r -action. In our specific case, the components for $i \in \Lambda$ act trivially – in what follows we will simply prove a more general assertion, regarding M_Λ and $M_{\Lambda'}$ as arbitrary finitely generated unit modules in $D_{\mathrm{fgu}}(A)^{h\Gamma}$ (using Kashiwara’s lemma).

Since N is invertible in $\mathbb{Z}_p$, for any $M \in D_{\mathrm{fgu}}(A)^{h\Gamma}$ the homotopy fixed points of its pullback are computed by taking the degree zero summand in the decomposition $A_N = \bigoplus_{\alpha \in (\mathbb{Z}/N\mathbb{Z})^r} A \cdot S^\alpha$. The projection onto this summand is Γ -equivariant. It is also Frobenius equivariant, since Frobenius permutes the summands by $\alpha \mapsto p\alpha$ and $(p, N) = 1$. The extension/restriction adjunction (in Frobenius modules) and preservation of limits by $\mathrm{RHom}(M_\Lambda, -)$ (to move $h\mu_N^r$ inside) therefore give

$$\begin{aligned} \mathrm{RHom}(f^*M_\Lambda, f^*M_{\Lambda'})^{h\mu_N^r} &\simeq \mathrm{RHom}\left(M_\Lambda, (A_N \widehat{\otimes}_A^L M_{\Lambda'})^{h\mu_N^r}\right) \\ &\simeq \mathrm{RHom}(M_\Lambda, M_{\Lambda'}). \end{aligned}$$

This proves the second claim. $\square$

We may then deduce the main theorem.

Theorem 5.29. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $W(k)[\zeta_p]$. Let $D_{\mathrm{cons}, D}^{(b), \mathrm{pKum}}((X, \mathcal{M})_\eta, \mathbb{Z}_p)$ denote the full subcategory of p -Kummer constructible sheaves in $D_{\mathrm{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbb{Z}_p)$. Then*

$$T_{\mathrm{\acute{e}t}} : D_{\mathrm{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge)^{\mathrm{op}} \xrightarrow{\sim} D_{\mathrm{cons}, D}^{(b), \mathrm{pKum}}((X, \mathcal{M})_\eta, \mathbb{Z}_p)$$

is a symmetric monoidal equivalence.

Proof. By Lemma 3.4 and strict étale descent (for the source Theorem 4.14 and for the target Lemma 4.24), we may first work on a framed log affine $(\mathrm{Spf} R, \mathcal{M})$.

In this case, we apply Lemma 5.17. We use the generators from Lemma 5.18 and Lemma 5.23, which correspond (up to dimension shifts, which do not matter for the argument) by combining Corollary 5.20 and Proposition 5.21.

We have now checked all the hypotheses:

- Both source and target are idempotent complete, and $T_{\mathrm{\acute{e}t}}$ is exact. This is explained already immediately after Lemma 5.17.
- The generators of the source and target correspond under $T_{\mathrm{\acute{e}t}}$, as explained above.
- The comparisons on RHoms are done by Theorem 5.27.

Thus $T_{\text{ét}}$ is an equivalence.

We further claimed that it is a symmetric monoidal equivalence. The fact that the equivalence is symmetric monoidal can also be tested locally, where via Lemma 4.30 it reduces to symmetric monoidality in Theorem 2.6. $\square$

We must also understand perverse t -exactness. We use a definition of the perverse t -structure on the Kummer-étale side of the correspondence which is similar to [BH22].

Definition 5.30. For each logarithmic stratum $j_\Lambda : (X, \mathcal{M})_{\Lambda, \eta} \hookrightarrow (X, \mathcal{M})_\eta$, put $d_\Lambda = \dim((X, \mathcal{M})_{\Lambda, \eta})$. The p -perverse t -structure of [Gab04] on the full subcategory of p -Kummer sheaves in $D_{\text{cons}, D}^{(b)}((X, \mathcal{M})_\eta, \mathbf{Z}_p)$ is defined by

$${}^p D^{\leq 0} = \{ \mathcal{F} \mid j_\Lambda^* \mathcal{F} \in D^{\leq -d_\Lambda}((X, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p) \text{ for every } \Lambda \},$$

$${}^p D^{\geq 0} = \{ \mathcal{F} \mid j_\Lambda^! \mathcal{F} \in D^{\geq -d_\Lambda}((X, \mathcal{M})_{\Lambda, \eta}, \mathbf{Z}_p) \text{ for every } \Lambda \}.$$

We write ${}^p H^i$ for its cohomology functors. The p^+ -perverse t -structure is

$${}^{p^+} D^{\leq 0} = \{ \mathcal{F} \in {}^p D^{\leq 1} \mid {}^p H^1(\mathcal{F}) \text{ is locally bounded } p\text{-power torsion} \},$$

with the ≥ 0 part simply given by

$${}^{p^+} D^{\geq 0} = \{ \mathcal{F} \mid \mathcal{F} // p \in {}^p D^{\geq 0}((X, \mathcal{M})_\eta, \mathbf{F}_p) \}.$$

One may check these t -structures restrict to the p -Kummer subcategory (i.e. that perverse truncations of p -Kummer sheaves are p -Kummer). The argument of [BH22, Construction 4.3 and Proposition 4.6], applied to the defining bounds after j_Λ^* and $j_\Lambda^!$, gives the following reduction criteria: for every $n \geq 1$, derived reduction modulo p^n is left t -exact for p^+ -perversity. Moreover, $\mathcal{F} \in {}^{p^+} D^{\geq 0}$ if and only if $\mathcal{F} // p \in {}^p D^{\geq 0}((X, \mathcal{M})_\eta, \mathbf{F}_p)$.

Using the equivalence with the essential image, the main content of the following proposition can be reduced to the analogous perverse t -exactness of the Emerton-Kisin correspondence.

Proposition 5.31. *Assume that $(X, \mathcal{M})_\eta$ is pure of dimension d . The equivalence of Theorem 5.29 with its essential image, shifted by $[d]$, is t -exact for the standard t -structure on the source and the p^+ -perverse t -structure of Definition 5.30 on the target.*

Proof. The source t -structure is strict étale local by Corollary A.19, and the same holds for the bounds in Definition 5.30. We may therefore work on a framed log affine.

We first prove the claim modulo p and therefore use the base $\overline{A} = \overline{R}^\square((q-1))$. We may consider $M \in D_{\text{fgu}}(\overline{A})^{h\Gamma}$, using Theorem 4.7.

For $\Lambda \subseteq \{1, \dots, r\}$, as before put

$$\overline{A}_\Lambda = (\overline{A}/(T_i)_{i \in \Lambda}) [1/T_j]_{j \notin \Lambda, j \leq r}$$

and write M_Λ for the derived base change $M \otimes_{\overline{A}}^L \overline{A}_\Lambda$.

Let $j_\Lambda : X_{\eta, \Lambda} \hookrightarrow (X, \mathcal{M})_\eta$ be the corresponding stratum, of dimension $d - |\Lambda|$. Lemma 4.27 and the proof of Proposition 4.28 show that each $H^i(M_\Lambda)$ is finite projective over $\overline{A}_\Lambda$. Compatibility with restriction, Lemma 4.30, and the contravariant Katz correspondence therefore yield

$$j_\Lambda^* T_{\text{ét}, \mathbf{F}_p}(M)[d] \in D^{\leq -(d-|\Lambda|)} \iff M_\Lambda \in D^{\geq -|\Lambda|}.$$

We claim that $M \in D^{\geq 0}$ if and only if $M_\Lambda \in D^{\geq -|\Lambda|}$ for every Λ , where we use the ordinary t -structure on $D_{\text{fgu}}(\overline{A})^{h\Gamma}$. Applying the coconnectivity criterion of [ATJLS10, Lemma 3.1] along the coordinate stratification, using $i_\Lambda^! M \simeq M_\Lambda[-|\Lambda|]$, gives the claim.

Definition 5.30 now gives

$$T_{\text{ét}, \mathbf{F}_p}(M)[d] \in {}^p D^{\leq 0} \iff M \in D^{\geq 0}.$$

Thus the contravariant equivalence exchanges coconnectivity on the source with perverse connectivity on the target. By orthogonality, it also exchanges connectivity with perverse coconnectivity, proving mod p perverse t -exactness.

To deduce the general claim, we see $M \in D^{\leq 0}$ if and only if $M//p \in D^{\leq 0}$ by derived Nakayama. On the Kummer-étale side we have analogous reduction criteria explained in Definition 5.30, so for the ≤ 0 part we may reduce to the mod p assertion. Orthogonality then proves the general claim. $\square$

It is also possible to extract ordinary perverse sheaves from the correspondence. We warn that extracting ordinary constructible sheaves in the derived sense cannot work without altering the crystal side of the correspondence, as these are not a full subcategory of the derived category of Kummer-étale sheaves (simply because Kummer-étale cohomology and étale cohomology differ).

Remark 5.32. Following [KL16, Theorem 7.5.6], call $\mathcal{E} \in D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$ corresponding to an object in the ordinary heart of constructible sheaves *effective* if, on every framed strict étale chart, its associated (φ, Γ) -module M satisfies the following condition for every boundary coordinate T_i : the i th $\mathbf{Z}_p(1)$ -factor acts trivially on

$$(A/(T_i)) \otimes_A^L M$$

in the derived category. This condition is strict étale local and independent of the framing. Theorem 5.29 gives an equivalence

$$T_{\text{ét}} : D_{\text{fgu}}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)^{\text{eff, op}} \xrightarrow{\sim} D_{\text{cons}, D}^\heartsuit(X_{\eta, \text{proét}}, \mathbf{Z}_p).$$

where on the target the ordinary t -structure is used.

In the derived setting, it seems plausible that one could take the “effective part” of an RHom to obtain the derived category of constructible sheaves along D .

We remark that trivial modifications of the previous arguments allow one to deduce the theorem over a slightly more general base. In particular our restriction to $W(k)[\zeta_p]$ up until this point is mainly for expository reasons to avoid tracking additional variables and cases throughout.

Corollary 5.33. *Theorem 5.29 and all preceding results stated over $W(k)[\zeta_p]$ admit corresponding variants over $W(k)[\zeta_{p^\ell}]$ for every $\ell \geq 0$.*

Proof. Corollary 3.21 is the only nontrivial change needed in the argument, which covers these cases. With this, one can deduce Theorem 4.14 verbatim (as it uses nothing about Γ). The following arguments are insensitive to the change in prismatic ideal or Γ , except for Proposition 5.9; however there the reduction is to vanishing of a cohomology group, so changing Γ from $(1 + p\mathbf{Z}_p) \ltimes \mathbf{Z}_p(1)^n$ to $(\mu_{p-1} \times (1 + p\mathbf{Z}_p)) \ltimes \mathbf{Z}_p(1)^n$ does not change the result by using Hochschild-Serre. $\square$

Remark 5.34. The entire local theory using (φ, Γ) -modules seems to have no obstruction to working over $\mathbf{A}_K$, but unfortunately the globalization step does not appear to work. One would want to have a prism $(\mathbf{A}_K^+, \varphi^n([p]_q))$ which satisfies

$$\mathbf{A}_K^+[1/\varphi^n([p]_q)]_p^\wedge \simeq \mathbf{A}_K.$$

However it seems such a prism cannot exist in general without dropping some desirable properties. See [Wat26, Appendix 2, Proposition 23] for more details. The argument there showing nonexistence uses that the mod p reduction should be E_K^+ , the ring of integers of the field of norms. It is still possible that $\mathbf{A}_K^+$ exists by dropping this condition, but this seems unnatural to do.

The following explicit (φ, Γ) -module gives a constructible sheaf with nontrivial gluing between the two strata.

Example 5.35. Let $(X, \mathcal{M}) = (\mathrm{Spf} \mathbf{Z}_p\langle T \rangle, \mathbf{N})$ with log structure induced by $D = \{T = 0\}$, and let $j : U_\eta = X_\eta \setminus D_\eta \hookrightarrow (X_\eta, \mathbf{N})$. Put $\bar{R} = \mathbf{F}_p\langle T \rangle$ and $A = \bar{R}^\square((q-1))$, using the framing given by T .

Following [Bri19], let $f : \mathcal{E} \rightarrow \mathbb{D} = X_\eta$ be the projectivization of the Tate family with parameter $\tilde{q} = p^p T$, given by

$$v^2 + uv = u^3 - b_2(\tilde{q}) - b_3(\tilde{q}),$$

where b_2 and b_3 are the standard coefficients for the Tate curve in $\tilde{q}\mathbf{Z}_p[[\tilde{q}]]$.

It is smooth away from $T = 0$, where it becomes a nodal cubic. The ordinary Zariski constructible sheaf

$$\mathcal{F} = R^1 f_* \mathbf{F}_p$$

therefore has rank two on U_η and rank one at the origin. On U_η the restriction of $\mathcal{F}$ is described by $\underline{\mathrm{Hom}}_{\mathbf{F}_p}(\mathcal{E}[p], \mathbf{F}_p)$. As explained in [Bri19, §3.1] one may describe $\mathcal{E}[p]$ on U_η as an extension of $\mathbf{F}_p$ by μ_p via the Kummer class $[\tilde{q}]$, and since $\tilde{q} = p^p T$ this is also the nonzero Kummer class of T . That is, $[\tilde{q}] = [T]$ in $H_{\text{ét}}^1(U_\eta, \mu_p)$ as classes obtained via the Kummer map $\mathcal{O}(U_\eta)^\times \rightarrow H_{\text{ét}}^1(U_\eta, \mu_p)$ since the kernel consists of p th powers. Brinon also explains the explicit formula for the Galois action, using this Kummer class $[\tilde{q}]$ for the Tate parameter.

Globally, the sheaf $\mathcal{F}$ is a nonsplit extension

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{F} \longrightarrow j_! \mathbf{F}_p(-1) \longrightarrow 0.$$

Here we deduce it is nonsplit by dualizing the previous description of $\mathcal{E}[p]$ as a nonsplit extension corresponding to $[\tilde{q}] \neq 0$ on the open stratum.

The corresponding (φ, Γ) -module has underlying module

$$M = A[1/T]e_1 \oplus Ae_2.$$

The submodule $A[1/T]e_1$ and quotient Ae_2 correspond respectively to $j_!\mathbf{F}_p(-1)$ and $\mathbf{F}_p$, in accordance with the contravariance of $T_{\text{ét}}$. Frobenius fixes both basis vectors. For $\gamma = (a_\gamma, b_\gamma) \in \Gamma = \mathbf{Z}_p^\times \ltimes \mathbf{Z}_p(1)$, we use the matrix

$$R_\gamma = \begin{pmatrix} \bar{a}_\gamma & \bar{b}_\gamma \\ 0 & 1 \end{pmatrix}$$

where the bars denote reduction modulo p to give the action of Γ . This means for a_γ , in the decomposition $\mathbf{Z}_p^\times = \mathbf{F}_p^\times \times (1 + p\mathbf{Z}_p)$ we extract only the $\mathbf{F}_p^\times$ component as $\bar{a}_\gamma$. Note $R_\gamma \cdot R_\delta = R_{\gamma\delta}$ as needed.

This realizes $\mathcal{F}$ (applying $T_{\text{ét}, \mathbf{F}_p}$).

6. PUSHFORWARD COMPATIBILITY

6.1. Six functors and Laurent stacks. The category $\mathbf{D}_\bullet((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta)$ is large enough that the six-functor results imply their Laurent analogues. For ordinary formal schemes, this is carried out in [Kip26b, Kip26a] by constructing an analytic stack $X^{\Delta, \blacksquare}$. We construct the logarithmic analogue and compare it with our site-theoretic category, and deduce several useful six functors inputs that we will use.

Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$, where K is a complete discretely valued field of mixed characteristic $(0, p)$ with perfect residue field k . As explained in more detail in Definition A.1, the ordinary log prismatic stack is defined by the fiber product

$$\begin{array}{ccc} (X, \mathcal{M})^\Delta & \longrightarrow & X^\Delta \\ \downarrow & & \downarrow \\ \mathbf{Z}_p^\Delta \times \text{Log} & \longrightarrow & \text{Log}^\Delta \end{array}$$

where Log is the formal p -completion of Olsson's log stack [Ols03]. This is a relative prismatic stack over Log , viewed as a “ δ -stack”, so the constructions of [Kip26a] apply by base change.

In [Kip26a, Construction 2.2.7], the éid topology is defined. Roughly, one takes the topology generated by étale covers and integral descendable covers; as explained in [Kip26a, 2.2.5], integrality for formal schemes is tested on ordinary schemes, so for formal affines one checks it on rings modulo an ideal of definition. We will always use integrality in this sense.

Definition 6.1. For a formal prestack $\mathcal{Y}$ over $\mathbf{Z}_p^\Delta$, let $\text{sh}_{\text{éid}}\mathcal{Y}$ denote the éid sheafification of the associated presheaf on $\text{fSch}/\mathbf{Z}_p^{\text{aff}}$ from [Kip26a, Construction 2.2.7]. We define

$$\mathcal{Y}^\blacksquare := (-)_{\text{Kipp}}^\blacksquare(\text{sh}_{\text{éid}}\mathcal{Y}),$$

which we call informally the *analytic realization* where

$$(-)_{\text{Kipp}}^\blacksquare : (\mathbf{Z}_p^\Delta)_{\text{éid}} \longrightarrow \text{AnStack}/\mathbf{Z}_p^{\Delta, \blacksquare}$$

is the colimit-preserving left exact functor of [Kip26a, Theorem 2.3.3(A) and §3.6.1]. We note $(-)^\blacksquare$ preserves finite limits, and $(\text{Spf } B)^\blacksquare = \text{Spa}(B, B)$ for a bounded prism B .

Strictly speaking, this construction is not the same as in [Kip26a]; however this definition will be more convenient for us, as the log prismatic stack is a type of relative stack so we need a

version of analytification which also works there. It turns out, however, that this is the same as Kipp's construction. In [Kip26a, 3.1.9] the topos $\mathrm{Sh}(\mathrm{Spf}(\mathbf{Z}_p)_{\sqrt{\infty}, \acute{e}t})$ is introduced, or sheaves for the topology generated by infinite root covers and étale covers. The functor $\mathrm{Spf} R \mapsto \mathrm{Spa}(\Delta_R, \Delta_R)$ sends covers in this topology to effective epimorphisms of analytic stacks. It is shown that semiperfectoids form a basis of the topology $\mathrm{Spf}(\mathbf{Z}_p)_{\sqrt{\infty}, \acute{e}t}$, and so one obtains a colimit preserving functor

$$\mathrm{Sh}(\mathrm{Spf}(\mathbf{Z}_p)_{\sqrt{\infty}, \acute{e}t}) \longrightarrow \mathrm{AnStack}_{/\mathbf{Z}_p^{\Delta, \blacksquare}}$$

induced by $\mathrm{Spf} R \mapsto \mathrm{Spa}(\Delta_R, \Delta_R)$ on the semiperfectoid basis. This is fine in the non-logarithmic setting, but unfortunately in the logarithmic setting semiperfectoids are problematic (they may not even have maps to the log prismatic stack, as they often lack fine log structures).

Fortunately the two constructions are the same, although this is not obvious. What is required is to show that for an infinite root/étale cover $\mathrm{Spf} R \rightarrow X$ that

$$\mathrm{Spf} \Delta_R \longrightarrow X^{\Delta}$$

is an effective epimorphism in the éid topos. Then we may use the corresponding colimit presentation coming from the Čech nerve to present X^{Δ} in the éid topos, and then since both functors are colimit preserving we get the same colimit presentation in analytic stacks.

In fact we claim more generally that for an infinite root/étale cover $\mathrm{Spf} A \rightarrow \mathrm{Spf} B$ that after prismatization and éid sheafification we get an effective epimorphism in the éid topos. It suffices to check this on generating covers; for étale covers it's clear, so one must show

$$\mathrm{sh}_{\acute{e}id} V_{\infty}^{\Delta} \longrightarrow \mathrm{sh}_{\acute{e}id} V^{\Delta}$$

is an effective epimorphism where $V = \mathrm{Spf} \mathbf{Z}_p \langle t_i \rangle_{i \in I}$ and $V_{\infty} = \mathrm{Spf} \mathbf{Z}_p \langle t_i^{1/p^{\infty}} \rangle_{i \in I}$. In the affine case it is easy to see the éid sheafification is in fact redundant, because $W(-)$ is an éid sheaf in spectra by [Kip26a, 3.2.1] (in particular one may apply the argument in the easy affine case of [BL22b, Lemma 7.3]). The argument of [BL22b, Lemma 6.3] applies to see we get an effective epimorphism. Consider an R -point of V^{Δ} . One explicit cover $R \rightarrow S$ constructed there (for S -point lift of V_{∞}^{Δ}) is in fact an éid cover. Upon unwinding their proof one first constructs, after orienting the Cartier-Witt divisor as $(d) \rightarrow W(R)$ by localizing,

$$B = W(R) \langle x_i^{1/p^{\infty}} \rangle_{i \in I} \left\{ \frac{x_i - \tilde{t}_i}{d} \right\}_{(p, d)}^{\wedge}$$

where $\tilde{t}_i$ are lifts of t_i to $W(R)$ and $\delta(x_i^{1/p^n}) = 0$. Then $S = B \otimes_{W(R)} R$ (underived by their flatness result). Descendability follows from the proof of [Kip26b, Proposition 3.2.1.8]. For integrality, Lemma A.14 shows that $\pi_0(B // (p, d))$ is integral over the algebra obtained from $\pi_0(W(R) // (p, d))$ by adjoining compatible p -power roots of the images of the $\tilde{t}_i$. This algebra is itself integral over $\pi_0(W(R) // (p, d))$. Since the image of (p, d) in R is nilpotent, we deduce $R \rightarrow S$ is integral, hence an éid cover.

Thus, we use Definition 6.1 throughout. In particular, we may then safely define

$$\begin{array}{ccc} (X, \mathcal{M})^{\Delta, \blacksquare} & \longrightarrow & X^{\Delta, \blacksquare} \\ \downarrow & & \downarrow \\ (\mathbf{Z}_p^{\Delta} \times \mathrm{Log})^{\blacksquare} & \longrightarrow & \mathrm{Log}^{\Delta, \blacksquare} \end{array}$$

where the notation on the bottom is interpreted according to Definition 6.1, and thus includes $\text{\acute{e}id}$ sheafification. Finite (derived) limit preservation of $(-)^{\Delta, \blacksquare}$ shows one may also construct $(X, \mathcal{M})^{\Delta, \blacksquare}$ by applying analytic realization to the already existing formal stack over $\mathbf{Z}_p^{\Delta}$.

The logarithmic Hodge-Tate substack is the fiber product $(X, \mathcal{M})^{\Delta, \blacksquare} \times_{\mathbf{Z}_p^{\Delta, \blacksquare}} \mathbf{Z}_p^{\text{HT}, \blacksquare}$, where $\mathbf{Z}_p^{\text{HT}, \blacksquare}$ is as in [Kip26a]. Following [Kip26a, Construction 4.3.7 and §4.3.9], we write $(X, \mathcal{M})^{\mathcal{L}, \blacksquare}$ for the algebraic nonvanishing locus of the Hodge-Tate ideal, further base changed along $\text{colim}_n \text{Spa}(\mathbf{Z}/p^n) \rightarrow \text{Spa} \mathbf{Z}_p$. Thus, after pullback along an orientable prism $(B, (d))$, it is given by

$$\text{colim}_{n \geq 1} \text{Spa}((B//p^n)[1/d], B//p^n).$$

Since this stack is obtained from $X^{\Delta, \blacksquare}$ by base change, the relevant results of [Kip26a] apply to it. See [Kip26a, §1.1] for background on cohomological smoothness and weak cohomological properness in six functor formalisms.

Lemma 6.2. *Let $f : X \rightarrow Y$ be a morphism of analytic stacks, and suppose it is cohomologically smooth and weakly cohomologically proper. Then given a square*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\iota_X} & X \\ \downarrow f_Z & & \downarrow f \\ Z & \xrightarrow{\iota_Y} & Y \end{array}$$

the map $f_Z : X \times_Y Z \rightarrow Z$ is cohomologically smooth and weakly cohomologically proper. Moreover, $f_Z^! \mathcal{O}_Z \simeq \iota_X^ f^! \mathcal{O}_Y$.*

Proof. Weak cohomological properness and cohomological smoothness are stable under arbitrary base change by [Kip26a, Remark 1.1.8], so f_Z is again cohomologically smooth and weakly cohomologically proper. Put $\omega_f = f^! \mathcal{O}_Y$. For a cohomologically smooth map, ω_f is invertible, the natural map $\omega_f \otimes f^*(-) \rightarrow f^!(-)$ is an equivalence, and ω_f commutes with arbitrary base change (see [Kip26b, Definition 1.2.1.1]). Hence the square in the Lemma statement gives $\omega_{f_Z} \simeq \iota_X^* \omega_f$. Applying this to the unit object gives $f_Z^! \mathcal{O}_Z \simeq \omega_{f_Z} \simeq \iota_X^* f^! \mathcal{O}_Y$. $\square$

Lemma 6.3. *Let $f : X \rightarrow Y$ be a cohomologically smooth and weakly cohomologically proper morphism of analytic stacks, and put $\omega_f = f^! \mathcal{O}_Y$. Suppose further that*

$$Rf_! \simeq Rf_*$$

via the canonical map.

Then Rf_ preserves dualizable objects, and for every dualizable object $\mathcal{E} \in D_{\blacksquare}(X)$ there is a natural equivalence*

$$(Rf_* \mathcal{E})^{\vee} \simeq Rf_*(\mathcal{E}^{\vee} \otimes \omega_f).$$

Proof. For every $\mathcal{E}' \in D_{\blacksquare}(Y)$, one has

$$\underline{\text{RHom}}_Y(Rf_* \mathcal{E}, \mathcal{E}') \simeq Rf_* \underline{\text{RHom}}_X(\mathcal{E}, f^! \mathcal{E}') \simeq Rf_*(\mathcal{E}^{\vee} \otimes \omega_f) \otimes \mathcal{E}'.$$

The first isomorphism follows from $\underline{\text{RHom}}_Y(Rf_! \mathcal{E}, \mathcal{E}') \simeq Rf_* \underline{\text{RHom}}_X(\mathcal{E}, f^! \mathcal{E}')$ (which one can check by showing maps into each are the same, and using the tensor/Hom adjunction and

the projection formula) and $Rf_! \simeq Rf_*$, while the second is just using the formula $f^!(-) \simeq \omega_f \otimes Lf^*(-)$.

Thus $Rf_*\mathcal{E}$ is dualizable with dual $Rf_*(\mathcal{E}^\vee \otimes \omega_f)$. $\square$

Lemma 6.4. *Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a strict log smooth proper morphism of log p -adic formal schemes satisfying Assumption 3.1 over $\mathcal{O}_K$ of pure relative dimension d . Then the induced maps on $(X, \mathcal{M}_X)^{\Delta, \blacksquare}$ and $(X, \mathcal{M}_X)^{\mathcal{L}, \blacksquare}$ are cohomologically smooth and weakly cohomologically proper, and on both stacks one has*

$$f^!\mathcal{O} \simeq \mathcal{O}\{d\}[2d].$$

Moreover $Rf_! \simeq Rf_*$ via the canonical map.

Proof. We first will argue this without log structures, and deduce the actual statement of the lemma as a formal consequence.

The underlying map of formal schemes is smooth and proper by strictness. Theorem 4.1.3(C) (using Example 4.3.2 of [Kip26a] to deduce analogous claims for the prismaticization) therefore show that the induced map

$$X^{\Delta, \blacksquare} \longrightarrow Y^{\Delta, \blacksquare}$$

is cohomologically smooth and weakly cohomologically proper. Using Theorem 4.1.3(F) we also get $f^!\mathcal{O} \simeq \mathcal{O}\{d\}[2d]$. However, weak cohomological properness is insufficient to deduce the last claim that $Rf_! \simeq Rf_*$ via the canonical map: it only gives a noncanonical isomorphism between these. The main content of this lemma will be to fix this issue, using results from [Mik25].

In what follows we use $D_\blacksquare$ to denote the six functor formalism on analytic stacks. Let π_2 denote the second projection from $X^{\Delta, \blacksquare} \times_{Y^{\Delta, \blacksquare}} X^{\Delta, \blacksquare}$. Weak cohomological properness implies that f is $D_\blacksquare$ -prim, so [Mik25, Corollary 1.28] gives a codualizing object

$$\delta_f \simeq R\pi_{2,*} R\Delta_{f,!} \mathcal{O}$$

and a natural equivalence $Rf_!(\delta_f \otimes -) \xrightarrow{\sim} Rf_*$.

Thus it remains to trivialize δ_f canonically. Since $\pi_2 \circ \Delta_f = \text{id}$, it is enough to construct a canonical equivalence $R\Delta_{f,!} \simeq R\Delta_{f,*}$. Since $f : X \rightarrow Y$ is smooth and separated, the diagonal $X \rightarrow X \times_Y X$ on the formal schemes is a regular closed immersion and hence locally of finite presentation. The induced map Δ_f on $(-)^{\Delta, \blacksquare}$ is therefore $!$ -able by [Kip26a, Theorem 4.1.3(A)]. The argument will be to apply [Mik25, Lemma 1.50]¹¹, upon which the desired equivalence holds provided that Δ_f is, universally $D_\blacksquare(-)^*$ -locally on its target represented by affine proper morphisms (compare [Kip26a, §2.1.10] and the class in [Mik25, Definition 1.5(2)]).

We now check this is true. Since the functor $X \mapsto X^{\Delta, \blacksquare}$ preserves finite limits by [Kip26a, Construction 3.7.1], the target of Δ_f is $(X \times_Y X)^{\Delta, \blacksquare}$. Work on the semiperfectoid cover $\text{Spa}(\Delta_{R_0}, \Delta_{R_0})$ of this target from [Kip26a, Construction 3.7.1]. After taking a Zariski refinement, we can assume f is smooth of pure relative dimension d (as a map of formal schemes), so the diagonal is a regular immersion of codimension d . Thus, after this localization, the pullback of Δ_f on analytic stacks is induced by a derived quotient $S_0 = R_0 // (g_1, \dots, g_d)$. Choose a perfectoid ring R surjecting onto R_0 , lift the g_i to $f_i \in R$, and put $S = R // (f_1, \dots, f_d)$, again as a derived quotient. The setup is summarized by the following Cartesian squares:

¹¹See [Mik25, Definition 1.10] for why it applies to analytic stacks.

$$\begin{array}{ccc}
X^{\Delta, \blacksquare} & \xrightarrow{\Delta_f} & X^{\Delta, \blacksquare} \times_{Y^{\Delta, \blacksquare}} X^{\Delta, \blacksquare} \\
\uparrow & & \uparrow \\
\mathrm{Spa}(\Delta_{S_0}, \Delta_{S_0}) & \xrightarrow{\Delta_{f, R_0}} & \mathrm{Spa}(\Delta_{R_0}, \Delta_{R_0}) \\
\downarrow & & \downarrow \\
\mathrm{Spa}(\Delta_S, \Delta_S) & \longrightarrow & \mathrm{Spa}(\Delta_R, \Delta_R).
\end{array}$$

It is therefore enough to show that $\mathrm{Spa}(\Delta_S, \Delta_S) \longrightarrow \mathrm{Spa}(\Delta_R, \Delta_R)$ is affine proper. Indeed, by construction $\mathrm{Spa}(\Delta_{R_0}, \Delta_{R_0})$ is an effective epimorphism of analytic stacks, hence a universal $D_{\blacksquare}^*$ -cover. After a further Zariski refinement, the prismatic ideal is $I = (\tilde{d})$ and we may choose lifts $\tilde{f}_i \in \Delta_R$ of $f_i \in R \simeq \Delta_R/I$. One then has

$$\Delta_S \simeq \Delta_R \left\{ \frac{\tilde{f}_1}{\tilde{d}}, \dots, \frac{\tilde{f}_d}{\tilde{d}} \right\}_{\delta, (p, \tilde{d})}^{\wedge}.$$

Lemma A.14 and [Kip26a, Example 2.1.13] then show that $\mathrm{Spa}(\Delta_S, \Delta_S) \longrightarrow \mathrm{Spa}(\Delta_R, \Delta_R)$ is affine proper.

Thus we have verified necessary hypotheses for Δ_f , and hence obtain a natural equivalence $R\Delta_{f,!} \simeq R\Delta_{f,*}$. As explained at the start, this shows the claim without log structures.

Deducing the claim with log structures added is immediate using base change. Indeed using strictness,

$$(X, \mathcal{M}_X)^{\Delta, \blacksquare} \simeq X^{\Delta, \blacksquare} \times_{Y^{\Delta, \blacksquare}} (Y, \mathcal{M}_Y)^{\Delta, \blacksquare}.$$

The claimed result for $(X, \mathcal{M}_X)^{\Delta, \blacksquare}$ then follows from Lemma 6.2 and compatibility of $Rf_! \simeq Rf_*$ with arbitrary base change. For the Laurent stack $(X, \mathcal{M}_X)^{\mathcal{L}, \blacksquare}$, the claim again follows by base change. $\square$

To actually make use of these stacks, we must show that they recover solid Laurent crystals defined using the site.

Proposition 6.5. *For every log formal scheme $(X, \mathcal{M})$ satisfying Assumption 3.1, there is a natural equivalence*

$$\mathrm{an}_{(X, \mathcal{M})} : D_{\blacksquare}((X, \mathcal{M})^{\mathcal{L}, \blacksquare}) \xrightarrow{\sim} D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}).$$

Proof. First, the equivalence is local because both sides satisfy strict étale descent. For the target this is shown in the appendix in Corollary A.18. For the source, by construction the stack itself satisfies strict étale descent, because $X^{\Delta, \blacksquare}$ satisfies étale descent by using [Kip26a, Theorem 4.1.3(D)] (again using Example 4.3.2 of [Kip26a] to deduce analogous claims for the prismaticization).

Thus we may work on a framed log affine. In this case, there exists a choice of log Breuil-Kisin prism atlas $(A^+, I, \mathcal{N})^a$ for this framed log affine such that we obtain the presentation

$$(X, \mathcal{M})^{\Delta, \blacksquare} = \mathrm{colim}_n \mathrm{Spa}(A^{+, (n)}, A^{+, (n)})$$

proved in Proposition A.22. Base changing to $(X, \mathcal{M})^{\mathcal{L}, \blacksquare}$ we obtain an analogous presentation, but replace the terms $\mathrm{Spa}(A^{+, (n)}, A^{+, (n)})$ with $\mathrm{colim}_m \mathrm{Spa}(A^{+, (n)}/p^m[1/I], A^{+, (n)}/p^m)$.

In Definition A.11, the category $D_{\blacksquare}(\mathrm{colim}_m \mathrm{Spa}(A^{+, (n)}/p^m[1/I], A^{+, (n)}/p^m))$ is precisely what we assign to the log prisms $A^{+, (n)}$ appearing in the Čech nerve of $(A^+, I, \mathcal{N})^a$. Combining with Theorem A.15, we obtain

$$D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \lim_n D_{\blacksquare}(\mathrm{colim}_m \mathrm{Spa}(A^{+, (n)}/p^m[1/I], A^{+, (n)}/p^m)) \simeq D_{\blacksquare}((X, \mathcal{M})^{\mathcal{L}, \blacksquare}).$$

Thus we get an identification for each log Breuil-Kisin prism. This identification is also independent of choices. One can see this by either assembling together the identifications for every log Breuil-Kisin prism into a canonical one, or noting that a canonical functor exists already. Indeed, for some $(A, (d), \mathcal{N})^a$ in the strict log prismatic site, applying the functor $(-)^{\blacksquare}$ (Definition 6.1) to the map of Lemma A.4 gives

$$\mathrm{Spa}(A, A) \longrightarrow (X, \mathcal{M})^{\Delta, \blacksquare},$$

so passing to Laurent stacks yields a canonical functor

$$D_{\blacksquare}((X, \mathcal{M})^{\mathcal{L}, \blacksquare}) \longrightarrow D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$$

This we call $\mathrm{an}_{(X, \mathcal{M})}$. □

Lemma 6.6. *For every $(X, \mathcal{M})$ satisfying Assumption 3.1, there is a functorial symmetric monoidal equivalence*

$$\mathrm{an}_{(X, \mathcal{M})} : \mathrm{Perf}((X, \mathcal{M})^{\mathcal{L}, \blacksquare}) \xrightarrow{\sim} \mathrm{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}).$$

*If $(Y, \mathcal{N})$ satisfies the same hypothesis and $f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ is strict log smooth and proper, denote by $f_{\mathcal{L}}$ is the induced map of Laurent analytic stacks. Then the site pushforward $Rf_{\Delta, *}$ preserves Perf and*

$$\mathrm{an}_{(Y, \mathcal{N})} Rf_{\mathcal{L}, *} \simeq Rf_{\Delta, *} \mathrm{an}_{(X, \mathcal{M})}.$$

Proof. This is formal from the previous results. For the first point, we constructed the functor through pullbacks of stacks so it will be symmetric monoidal. For the second point, this follows by taking right adjoints and observing an has pullback compatibility.

Once we know this, preservation of Perf is true on the stack side by applying Lemma 6.4 and Lemma 6.3. □

We also obtain Poincaré duality, where we can obtain $f^!$ on the site side by transporting through Proposition 6.5.

Corollary 6.7. *Let $f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ be as in Lemma 6.4, and suppose both source and target satisfy Assumption 3.1. Then $f^! \mathcal{F} \simeq f^* \mathcal{F}\{d\}[2d]$ for $\mathcal{F} \in \mathrm{Perf}((Y, \mathcal{N})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$, and*

$$Rf_{\Delta, *} \mathcal{E} \simeq \underline{\mathrm{RHom}}_{\mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}}(Rf_*(\mathcal{E}^{\vee}\{d\}), \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}[-2d])$$

for every $\mathcal{E} \in \mathrm{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$.

Proof. Transporting Lemma 6.4 through Lemma 6.6 gives $f^! \mathcal{F} \simeq f^* \mathcal{F}\{d\}[2d]$. Lemma 6.3, together with the pushforward comparison in Lemma 6.6, gives

$$(Rf_{\Delta,*} \mathcal{E})^\vee \simeq Rf_{\Delta,*}(\mathcal{E}^\vee\{d\})[2d].$$

Dualizing, this equivalence gives the claim. $\square$

We also note that the Poincaré duality pairing is compatible with Frobenius: the identification $f^! \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge \simeq \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge\{d\}[2d]$ is Frobenius equivariant since it is upgraded to an identification of F-gauges in [Kip26a, Theorem 4.1.3(F)].

6.2. Pushforward compatibility. Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a strict log smooth proper morphism of pure relative dimension d between log p -adic formal schemes whose adic generic fibers have pure dimensions d_X and d_Y . We assume $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ satisfy Assumption 3.1 over one of the bases $\mathrm{Spf} W(k)[\zeta_{p^\ell}]$.

Let $d = d_X - d_Y$. In such a setup, define

$$Rf_+(-) := Rf_*(-)\{d\}[d].$$

We will show that Rf_+ induces a functor

$$Rf_+ : D_{\mathrm{fgu}}((X, \mathcal{M}_X)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_{\mathrm{fgu}}((Y, \mathcal{M}_Y)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge).$$

After this, the pushforward compatibility result is formal.

We first need some preliminaries on relative stacks. Let $(Y, \mathcal{M}_Y)$ be a framed log affine with associated logarithmic q -de Rham prism $(A^+, I, \mathcal{N})^a$, so $Y = \mathrm{Spf}(A^+/I)$, and let $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be strict smooth. By Proposition A.22, the map $\mathrm{Spa}(A^+, A^+) \rightarrow (Y, \mathcal{M}_Y)^{\Delta, \blacksquare}$ is a representable proper effective epimorphism. Define $((X, \mathcal{M}_X)/A^+)^{\Delta, \blacksquare}$ via the Cartesian square

$$\begin{array}{ccc} ((X, \mathcal{M}_X)/A^+)^{\Delta, \blacksquare} & \longrightarrow & (X, \mathcal{M}_X)^{\Delta, \blacksquare} \\ \downarrow f_{A^+} & & \downarrow \\ \mathrm{Spa}(A^+, A^+) & \xrightarrow{\rho_{A^+}} & (Y, \mathcal{M}_Y)^{\Delta, \blacksquare} \end{array}$$

where the bottom map is the effective epimorphism of Proposition A.22. We define $((X, \mathcal{M}_X)/A^+)^{\mathcal{L}, \blacksquare}$ by further base change over $\mathrm{Spa}(A^+, A^+)$ to

$$(\mathrm{Spf}(A^+/I)/A)^{\mathcal{L}, \blacksquare} = \mathrm{colim}_n \mathrm{Spa}((A^+/p^n)[1/I], A^+/p^n).$$

Equivalently, we replace all terms in the Cartesian square with their Laurent versions. The method of Proposition 6.5 identifies $\mathrm{Perf}(((X, \mathcal{M}_X)/A^+)^{\mathcal{L}, \blacksquare})$ with $\mathrm{Perf}(((X, \mathcal{M}_X)/A^+)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge)$, by locally using the Čech presentation of the stack induced by a Breuil-Kisin prism for X and identifying it with the site-theoretic one.

There is a natural Frobenius φ on $((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare}$ over φ_A on $(\mathrm{Spf}(A^+/I)/A)^{\mathcal{L}, \blacksquare}$. In what follows the superscript (φ) denotes base change along this φ_A . If f_A denotes the structure map to $(\mathrm{Spf}(A^+/I)/A)^{\mathcal{L}, \blacksquare}$, we use Fr or relative Frobenius to mean the induced map $\mathrm{Fr} = (\varphi, f_A) : ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare} \rightarrow ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare, (\varphi)}$.

The following lemma is an analogue of Cartier descent. There are some variants already existing, such as in [GLSQ23] (see especially Corollary 6.8, which is a bit similar to our setup), but they do not suffice for our applications.

Lemma 6.8. *In the above setting, relative Frobenius induces a fully faithful functor*

$$\mathrm{Fr}^* : \mathrm{Perf}(((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare, (\varphi)}) \longrightarrow \mathrm{Perf}(((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare}).$$

Proof. The projection formula immediately reduces the claim to proving that the unit $\mathcal{O} \rightarrow \mathrm{R}\mathrm{Fr}_* \mathcal{O}$ is an equivalence. Since we have restricted to perfect complexes, we don't need to check Fr is proper first to use this.

We may also work strict étale locally by descent of the source and target (this is again automatic from the claim for $(X, \mathcal{M}_X)^{\Delta, \blacksquare}$ by base change): thus we assume that X is a framed log affine relative to $(Y, \mathcal{M}_Y)$ with relative coordinates $T_1, \dots, T_d$. More precisely, we have

$$(X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y) \times_{\mathrm{Spf} W(k)[\zeta_{p^\ell}]} \widehat{\mathbf{G}}_m^d \longrightarrow \mathbb{T}_r^{n+d}$$

where $\widehat{\mathbf{G}}_m^d$ has a trivial log structure. Thus $(X, \mathcal{M}_X)$ inherits a framing, and has a corresponding log q -de Rham prism with ring B^+ and $B = B^+[1/(q-1)]_p^\wedge$. A straightforward variant of Proposition 3.13 gives a proper morphism $\rho : [\mathrm{AnSpec} B/\mathbf{Z}_p^d] \rightarrow ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare}$ with a fully faithful pullback where $\mathbf{Z}_p^d$ fixes A and $\gamma_i \cdot T_j = q^{p^\ell \delta_{ij}} T_j$ for the generators γ_i of $\mathbf{Z}_p^d$. For $\ell = 0$, we use the same chart with $I = ([p]_q)$, which still gives a cover over $W(k)$. In this case $\gamma_i \cdot T_j = q^{p \delta_{ij}} T_j$ and $\varphi(I) = ([p]_{q^p})$; the proof is similar for this case so we focus on $\ell \geq 1$.

We first observe that there is a commutative diagram¹²

$$\begin{array}{ccc} [\mathrm{AnSpec} B/\mathbf{Z}_p^d] & \xrightarrow{\rho} & ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare} \\ \downarrow F & & \downarrow \mathrm{Fr} \\ [\mathrm{AnSpec} B^{(\varphi)}/\mathbf{Z}_p^d] & \xrightarrow{\rho^{(\varphi)}} & ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare, (\varphi)}. \end{array}$$

Here $\rho^{(\varphi)}$ is the pullback along φ_A and F is induced by the relative Frobenius

$$B^{(\varphi)} := B \widehat{\otimes}_{A, \varphi_A} A \longrightarrow B.$$

Moreover, from the argument of Proposition 3.13 we have seen $\mathrm{R}\rho_* \mathcal{O} = \mathcal{O}$ and that ρ is proper. We also get $\mathrm{R}\rho_*^{(\varphi)} \mathcal{O} \simeq \mathcal{O}$ by proper base change.

Now the diagram is useful: from $\mathrm{R}\rho_* \mathcal{O} = \mathcal{O}$, equivalently we can compute the pushforward along the other path, first computing $F_* B$ and then applying $\mathrm{R}\rho_*^{(\varphi)}$. The first part is immediate, as we see

$$F_* B \simeq \bigoplus_{\beta \in \{0, \dots, p-1\}^d} B^{(\varphi)} \prod_i T_i^{\beta_i}.$$

where the $\mathbf{Z}_p^d$ action acts on T_i in the usual way by $\gamma_i \cdot T_j = q^{p^\ell \delta_{ij}} T_j$. Now we must compute the pushforward for each summand. For $\beta = 0$, we have already seen $\mathrm{R}\rho_*^{(\varphi)} \mathcal{O} \simeq \mathcal{O}$. Thus our task is to show the remaining summands $B^{(\varphi)} \prod_i T_i^{\beta_i} \in \mathrm{D}_\blacksquare([\mathrm{AnSpec} B^{(\varphi)}/\mathbf{Z}_p^d])$ are killed by $\mathrm{R}\rho_*^{(\varphi)}$.

¹²Here we abuse notation, because what we mean by $\mathrm{AnSpec} B$ is actually $\mathrm{colim}_n \mathrm{Spa}(B^+/p^n[1/(q-1)], B^+/p^n)$ which is not affine. However this is too cumbersome to write out each time, so we will use $\mathrm{AnSpec} B$ for this reason.

To actually compute this we will need to apply proper base change again. We let $B^{(1),\varphi}$ be the Frobenius pullback along φ_A of the first (Laurentized) term in the Čech nerve of $\mathrm{AnSpec} B \rightarrow ((X, \mathcal{M}_X)/A)^{\mathcal{L},\blacksquare}$. We will instead use the square

$$\begin{array}{ccc} [\mathrm{AnSpec} B^{(1),(\varphi)}/\mathbf{Z}_p^d] & \longrightarrow & \mathrm{AnSpec} B^{(\varphi)} \\ \downarrow & & \downarrow \\ [\mathrm{AnSpec} B^{(\varphi)}/\mathbf{Z}_p^d] & \xrightarrow{\rho^{(\varphi)}} & ((X, \mathcal{M}_X)/A)^{\mathcal{L},\blacksquare,(\varphi)}. \end{array}$$

We know already that the right vertical map is a proper effective epimorphism. We can deduce this using Proposition A.22. Thus proper base change reduces us to showing that

$$\mathrm{R}\Gamma_{\blacksquare}(\mathbf{Z}_p^d, B^{(1),(\varphi)} \otimes_{B^{(\varphi)}, p_1} B^{(\varphi)} \prod_i T_i^{\beta_i}) = 0$$

for $\beta \neq 0$. This is now a self-coproduct in the relative site¹³, so we may compute

$$B^{+, (1), (\varphi)} = B^{+, (\varphi)}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \left\{ \frac{u_1 - 1}{[p]_{q^{p^\ell}}}, \dots, \frac{u_d - 1}{[p]_{q^{p^\ell}}} \right\}_{\delta, (p, q-1)}^\wedge.$$

Here $\delta(u_i) = 0$ and $u_i = T_{i,1}/T_{i,0}$. In particular, the formula is nearly identical to Lemma 3.6, so one may run the same argument to embed

$$B^{+, (1), (\varphi)} \subset C(\mathbf{Z}_p^d, B^{+, (\varphi)})$$

such that the mod p reduction, as in Lemma 3.10, identifies with the functions

$$\left(\bigoplus_{\alpha \in \mathbf{N}^d} \overline{B}^{+, (\varphi)} (q^{p^\ell} - 1)^{|\alpha|} \binom{x}{\alpha} \right)_{(q-1)}^\wedge \subset C(\mathbf{Z}_p^d, \overline{B}^{+, (\varphi)}).$$

When we consider $B^{(1),(\varphi)} \otimes_{B^{(\varphi)}, p_1} B^{(\varphi)} \prod_i T_i^{\beta_i}$ this simply twists the $\mathbf{Z}_p^d$ action, instead making

$$\gamma_i \cdot f(x) = q^{p^\ell \beta_i} f(x + e_i)$$

where e_i is the i th basis vector in $\mathbf{Z}_p^d$ (and we use x for the full coordinate). Taking any $\beta_i \neq 0$, this shows directly $\mathrm{fib}(\gamma_i - 1)$ is zero because $\frac{\gamma_i - 1}{q^{p^\ell} - 1} = [\beta_i]_{q^{p^\ell}} + q^{p^\ell \beta_i} \frac{\Delta_i}{q^{p^\ell} - 1}$ is invertible, where Δ_i is the finite difference operator. This shows the vanishing modulo p which suffices by derived Nakayama. $\square$

Proposition 6.9. *Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a strict log smooth proper morphism between log formal schemes satisfying Assumption 3.1 over $W(k)[\zeta_{p^\ell}]$ with $\ell \geq 0$. For every $\mathcal{E} \in \mathrm{Perf}((X, \mathcal{M}_X)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p)$, the canonical map*

$$\varphi^* \mathrm{R}f_* \mathcal{E} \longrightarrow \mathrm{R}f_* \varphi^* \mathcal{E}$$

is an isomorphism.

¹³Later in Lemma A.6, we compute this first Čech term in the absolute case (which extends to the Laurent stacks by finite limit compatibility of $(-)^{\blacksquare}$ from Definition 6.1). The relative computation follows formally from the absolute case, since for example $\mathrm{Spf} B \times_{(X/A)^\Delta} \mathrm{Spf} B = (\mathrm{Spf} B^{(1)}) \times_{\mathrm{Spf} A^{(1)}} \mathrm{Spf} A$, where $A^{(1)}$ and $B^{(1)}$ are the self-coproducts for the absolute sites.

Proof. By Proposition 6.5 and proper base change from Lemma 6.4 we may reduce to the case where $(Y, \mathcal{M}_Y)$ is a framed log affine; note here we also use strict étale descent from Corollary A.18. In the $\ell = 0$ case the log q -de Rham is not a Breuil-Kisin prism; in this case it is still a cover so the argument is similar.

We may evaluate the canonical comparison morphism on a logarithmic q -de Rham chart $(A, I, M_{\mathrm{Spf} A})$ of $(Y, \mathcal{M}_Y) = (\mathrm{Spf} A/I, M_{\mathrm{Spf} A/I})$. Pullback along this chart is conservative (since it gives an effective epimorphism of the log prismatic stack, as in Proposition A.22), so we need only check the claim on the relative stack. Also, by strictness X is smooth over $\mathrm{Spf} A/I$. Writing f_A for the structure map $(X/A)^{\mathcal{L}, \blacksquare} \rightarrow \mathrm{AnSpec} A[1/I]_p^\wedge = (\mathrm{Spf} A/I/A)^{\mathcal{L}, \blacksquare}$, we have a commutative diagram

$$\begin{array}{ccccc} ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare} & \xrightarrow{\mathrm{Fr}} & ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare, (\varphi)} & \xrightarrow{\tilde{\varphi}_A} & ((X, \mathcal{M}_X)/A)^{\mathcal{L}, \blacksquare} \\ & \searrow f_A & \downarrow f_A^{(\varphi)} & & \downarrow f_A \\ & & \mathrm{AnSpec} A[1/I]_p^\wedge & \xrightarrow{\varphi_A} & \mathrm{AnSpec} A[1/I]_p^\wedge \end{array}$$

whose right square is Cartesian, so $\tilde{\varphi}_A$ is the base change of φ_A along f_A , and $\tilde{\varphi}_A \circ \mathrm{Fr} = \varphi$. Put $\mathcal{E}^{(\varphi)} = \tilde{\varphi}_A^* \mathcal{E}$. Proper base change and Lemma 6.8 give

$$\begin{aligned} \varphi_A^* Rf_{A,*} \mathcal{E} &\simeq Rf_{A,*}^{(\varphi)} \mathcal{E}^{(\varphi)} \\ &\xrightarrow{\sim} Rf_{A,*}^{(\varphi)} R\mathrm{Fr}_* \mathrm{Fr}^* \mathcal{E}^{(\varphi)} \simeq Rf_{A,*} \varphi^* \mathcal{E}. \end{aligned}$$

Thus the claim follows. $\square$

Theorem 6.10. *Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be strict log smooth proper, where the source and target satisfy Assumption 3.1 over $\mathrm{Spf} W(k)[\zeta_{p^\ell}]$. Then Rf_+ induces a functor*

$$Rf_+ : D_{\mathrm{fgu}}((X, \mathcal{M}_X)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge) \longrightarrow D_{\mathrm{fgu}}((Y, \mathcal{M}_Y)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge).$$

Proof. Since the finitely generated unit category is stable under shifts and twists we may test the assertion for Rf_* . We may also test the assertion modulo p by derived Nakayama, and further reduce to the case where the p -torsion input $\mathcal{E}$ lies in the heart. Then by Corollary 4.16, there is a coherent root $\mathcal{E}_0$ of $\mathcal{E}$. But then since $Rf_* \simeq Rf_!$ canonically in this situation on the Laurent stacks and Proposition 6.5, the pushforward commutes with colimits. In particular,

$$Rf_*(\mathrm{colim}_n \varphi^{*n} \mathcal{E}_0) \simeq \mathrm{colim}_n Rf_* \varphi^{*n} \mathcal{E}_0.$$

By Proposition 6.9 we see this is $\mathrm{colim}_n \varphi^{*n} Rf_* \mathcal{E}_0$. Using Lemma 6.6, we see $Rf_* \mathcal{E}_0$ is perfect, which proves the result is finitely generated unit. $\square$

Theorem 6.11. *Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a strict log smooth proper morphism of pure relative dimension d where both satisfy Assumption 3.1 over $\mathrm{Spf} W(k)[\zeta_{p^\ell}]$ whose log adic generic fibers are pure of dimensions d_X and d_Y respectively. Let D_X and D_Y be horizontal normal crossings divisors inducing M_X and M_Y . There is a commutative diagram*

$$\begin{array}{ccc}
D_{\text{fgu}}((X, \mathcal{M}_X)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge)^{\text{op}} & \xrightarrow{T_{\text{ét}, X}[d_X]} & D_{\text{cons}, D_X}^{(b)}((X, \mathcal{M}_X)_\eta, \mathbf{Z}_p) \\
\downarrow Rf_*\{d\}[d] & & \downarrow Rf_{\eta,*} \\
D_{\text{fgu}}((Y, \mathcal{M}_Y)_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge)^{\text{op}} & \xrightarrow{T_{\text{ét}, Y}[d_Y]} & D_{\text{cons}, D_Y}^{(b)}((Y, \mathcal{M}_Y)_\eta, \mathbf{Z}_p).
\end{array}$$

Proof. Theorem 6.10 and Corollary 4.18 allow us to restrict the pushforward and pullback functors to the finitely generated unit categories. For $\mathcal{E}$ on X and $\mathcal{F}$ on Y , transporting Lemma 6.4 through Proposition 6.5 and using the projection formula give

$$\begin{aligned}
\text{RHom}(Rf_+\mathcal{E}[d], \mathcal{F}) &\simeq \text{RHom}(Rf_*(\mathcal{E}\{d\}[2d]), \mathcal{F}) \\
&\simeq \text{RHom}(\mathcal{E}\{d\}[2d], f^!\mathcal{F}) \simeq \text{RHom}(\mathcal{E}, Lf^*\mathcal{F}).
\end{aligned}$$

Thus $Rf_+[d]$ is left adjoint to Lf^* . The units and counits in this calculation are Frobenius compatible (see the remark after Corollary 6.7), so the adjunction persists after applying $(-)^{\varphi=1}$ to all categories in sight.

The functor $Rf_{\eta,*}$ also preserves p -Kummer constructible complexes, so the adjunction $Lf_\eta^* \rightleftarrows Rf_{\eta,*}$ restricts to these categories. To see this, first observe proper smooth pushforward preserves lissity on each logarithmic stratum. It is easy to check the p -Kummer condition is preserved as well: one may use a spectral sequence to understand the prime-to- p log inertia on the pushforward at a geometric stalk.¹⁴

We already saw that $Rf_+[d]$ is left adjoint to Lf^* . Pullback compatibility of $T_{\text{ét}}$ yields

$$T_{\text{ét}, X}^{-1} \circ Lf_\eta^* \rightleftarrows T_{\text{ét}, Y} \circ (Rf_+[d]).$$

Note here to transport the adjunction we are using the equivalence in 5.29 and that $Rf_+[d]$ preserves the crystal side and that Lf_η^* preserves p -Kummer sheaves.

Uniqueness of right adjoints therefore gives

$$T_{\text{ét}, Y}(Rf_+\mathcal{E}[d]) \simeq Rf_{\eta,*}T_{\text{ét}, X}(\mathcal{E}).$$

Here we have used the previous observation that $Rf_{\eta,*}$ preserves p -Kummer constructible sheaves. Since $d_X = d_Y + d$ this yields the asserted commutative diagram. $\square$

While we have primarily been interested in bases such as $W(k)[1/p]$, we remark that it is possible to also work over a fairly general perfectoid base.

Remark 6.12. Let K be a characteristic zero perfectoid field containing a compatible system of p -power roots of unity. We introduce a new prism $\mathbf{A}_{\text{inf}, R}^\square$, similar to the construction in [MW25]. For $R = \mathcal{O}_K\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$ we use the log prism

$$(\mathbf{A}_{\text{inf}}(\mathcal{O}_K)\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle_{(p, I)}, I, \mathcal{M}_{V(T_1 \dots T_r)})^a$$

where $I = \ker(\theta : \mathbf{A}_{\text{inf}}(\mathcal{O}_K) \rightarrow \mathcal{O}_K)$. For a general framed log affine over $\mathcal{O}_K$, it has a strict étale framing $\square$ to $\text{Spf } \mathcal{O}_K\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$ with the log structure $\mathcal{M}_{V(T_1 \dots T_r)}$. Similar to the construction of the log q -de Rham prism we may lift along the framing to obtain the desired log

¹⁴If one uses a spectral sequence to argue this, it is important to note invariants for the prime-to- p log inertia are exact. In particular it will again act trivially on extensions, which is why on the abutment of the spectral sequence it will still act trivially if it does on the E_2 page.

prism with ring $\mathbf{A}_{\text{inf},R}^\square$. We note the mod p reduction of $\mathbf{A}_{\text{inf},R}^\square[1/I]_p^\wedge$ is regular F -finite (it will be an affinoid algebra over $K^\flat$ étale over $K^\flat\langle T_1, \dots, T_r, T_{r+1}^\pm, \dots, T_n^\pm \rangle$).

This prism is equipped with a $\mathbf{Z}_p(1)^n$ action, and all arguments in the appendix for Breuil-Kisin atlases (indeed as this is a Breuil-Kisin atlas) readily adapt to this prism. In particular, all results thus far admit a simpler variant where we simply ignore the $\mathbf{Z}_p^\times$ component of Γ in the argument. Indeed the arguments which deal with something beyond the toric part of the action (so they do not adapt word for word) are Proposition 5.9 and Lemma 3.12. In the first we may make the same reduction to mod p vector bundle coefficients, and then in calculating the cohomology we can skip $\mathbf{A}_{\text{perf}}^{\circ, \text{cyc}}$. The method is identical, with the second helper lemma not being needed. In Lemma 3.12 the argument is identical, just omitting the $\eta_{0,m}$ from the basis. For the toric component, put $q = [(1, \zeta_p, \zeta_{p^2}, \dots)^{1/p}]$, so $I = ([p]_q)$ and $\gamma_i(T_j) = q^{p\delta_{ij}}T_j$. One may also argue as in Lemma 6.8 that the toric part of Γ action acts by $(\gamma_i - 1)\eta_\alpha = (q - 1)u_i\eta_{\alpha'}$ where α' is the same multi-index but with the i th position lowered by one, and negative indices mean zero. In particular the same argument also shows the desired cohomology calculation. Elsewhere, arguments adapt verbatim.

APPENDIX A. LOG PRISMATIC STACKS

A.1. The log prismatic stack. In this appendix by formal stacks we mean accessible fpqc sheaves in groupoids (or more generally spaces) on animated p -nilpotent rings.

We denote by Log the p -completion of Olsson's stack of fine log structures over $\mathbf{Z}_p$ [Ols03, Theorem 1.1]. For every fine monoid P , we have an associated toric stack

$$\mathcal{A}_P = [\text{Spf } \mathbf{Z}_p\langle P \rangle / \text{Spf } \mathbf{Z}_p\langle P^{\text{gp}} \rangle].$$

These $\mathcal{A}_P$ cover Log by [Ols03, Corollary 5.25]. One may check that these, and in fact Log , are formal δ -stacks in the sense of [AKN26, Construction 10.20].

The general philosophy behind the log prismatic stack is easy to motivate, the principal idea being that the data of a log formal scheme X with a fine log structure amounts to a map $X \rightarrow \text{Log}$. Moreover in this interpretation many logarithmic cohomology theories $H_{\log}(-)$ can often be produced by taking a relative version $H(X/\text{Log})$ using the ordinary cohomology theory $H(-)$. For example, logarithmic de Rham cohomology can be recovered this way. In [AKN26] they define prismatic stacks relative to δ -rings, but more generally, if $X \rightarrow \mathcal{S}$ is a morphism to a formal δ -stack, we may write

$$(X/\mathcal{S})^\Delta := X^\Delta \times_{\mathcal{S}^\Delta} (\mathcal{S} \times \mathbf{Z}_p^\Delta)$$

for its *relative prismaticization*. The point we explain next is what the map $\mathbf{Z}_p^\Delta \times \mathcal{S} \rightarrow \mathcal{S}^\Delta$ is.

We will use this construction with $\mathcal{S} = \text{Log}$, which we explain in more detail. We must specify a map $\text{Log} \times \mathbf{Z}_p^\Delta \rightarrow \text{Log}^\Delta$, as the other map is simply induced by functoriality of prismaticization. Let R be a p -nilpotent ring and let $\mathcal{M}_R \in \text{Log}(R)$. Regarding Witt vectors as the cofree δ -ring and using the δ -stack structure on Log gives

$$\text{Log}(R) \simeq \text{Map}_\delta(\text{Spf } W(R), \text{Log}) \longrightarrow \text{Map}(\text{Spf } W(R), \text{Log}).$$

Let $\mathcal{M}_{W(R)}$ be the resulting log structure. If $P \rightarrow \mathcal{M}_R$ is a fine chart with structure map $\alpha : P \rightarrow R$, then locally

$$\mathcal{M}_{W(R)} = \left(P \xrightarrow{m \mapsto [\alpha(m)]} W(R) \right)^a.$$

We then define the map as

$$\pi_{\text{Log}}(\mathcal{M}_R, I \rightarrow W(R)) = (I \rightarrow W(R), \mathcal{M}_{W(R)}|_{\text{Spec}(W(R)//I)}) .$$

Then fpqc sheafification yields $\pi_{\text{Log}} : \text{Log} \times \mathbf{Z}_p^\Delta \rightarrow \text{Log}^\Delta$. We remark here that $W(R)//I$ has a slightly different meaning, explained in [BL22b, §2.1].

Definition A.1. Let $(X, \mathcal{M})$ be a bounded fs log formal scheme. Then there is a map $X \rightarrow \text{Log}$ induced by the log structure. We define $(X, \mathcal{M})^\Delta$ to be the fiber product

$$\begin{array}{ccc} (X, \mathcal{M})^\Delta & \longrightarrow & X^\Delta \\ \downarrow & & \downarrow \\ \text{Log} \times \mathbf{Z}_p^\Delta & \xrightarrow{\pi_{\text{Log}}} & \text{Log}^\Delta. \end{array}$$

This is the log prismatic stack introduced by Olsson in [Ols23].¹⁵

Throughout this appendix, we work in the situation of Assumption 3.1 over $\mathcal{O}_K$, where $K/W(k)[1/p]$ is a finite extension.

Strict étale locally, every such log formal scheme is a framed log affine, for which there is a logarithmic Breuil-Kisin atlas. We define this terminology as follows.

Definition A.2. Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$. A logarithmic Breuil-Kisin prism for $(X, \mathcal{M})$ is a log prism $(A, I, M_{\text{Spf } A})^a$ such that

$$(\text{Spf } A/I, M_{\text{Spf } A/I}) \longrightarrow (X, \mathcal{M})$$

is a strict open immersion of log p -adic formal schemes.

A logarithmic Breuil-Kisin atlas for $(X, \mathcal{M})$ means this is an isomorphism.

Lemma A.3. Let $(X, \mathcal{M}) = (\text{Spf } R, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$. Then there is a logarithmic Breuil-Kisin prism $(A, I, M_{\text{Spf } A})^a$ which is an atlas for $(X, \mathcal{M})$.

Proof. This construction is also explained in [Ino25, Construction 5.10], but we explain it again here for convenience of the reader.

First suppose $K/W(k)[1/p]$ is finite with residue field k . Write $R \simeq R_0 \widehat{\otimes}_{W(k)} \mathcal{O}_K$ by the remark after Definition 3.2.

We first review what happens for a point. Choose a uniformizer π of $\mathcal{O}_K$ with Eisenstein polynomial $E(u)$, and give $\mathfrak{S} = W(k)[[u]]$ the δ structure with Frobenius $u \mapsto u^p$ (with the Witt vector Frobenius on $W(k)$). This makes $(\mathfrak{S}, (E(u)))$ a Breuil-Kisin prism for $\mathcal{O}_K$. When $\mathcal{O}_K = W(k)[\zeta_{p^\ell}]$ as in our applications, we may use the q -de Rham prism $(W(k)[[q-1]], (\varphi^{\ell-1}([p]_q)))$ with $q \mapsto q^p$ (for $\ell \geq 1$); this is also a Breuil-Kisin prism under our definition. We discuss a general construction compatible with the first case, but a similar construction works in the second case.

¹⁵We note that the definition of this stack was also considered by Koshikawa.

For a framed log affine, give

$$\mathbb{T}_r^n = \mathrm{Spf} W(k) \langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1} \rangle$$

the Frobenius lift induced by $T_i \mapsto T_i^p$. Since R_0 is p -completely étale over its ring of functions, we may lift Frobenius uniquely to R_0 . We then set

$$A = (R_0 \otimes_{W(k)} \mathfrak{S})_{(p, E(u))}^\wedge$$

with prismatic ideal $E(u)$ and the chart $\mathbb{N}^r \rightarrow A$ sending the i th generator to T_i and $\delta_{\log}(e_i) = 0$, giving a bounded fine log prism. One may check

$$(\mathrm{Spf} A/I, M_{\mathrm{Spf} A/I}) \simeq (X, \mathcal{M}).$$

□

A.2. Quasicoherent sheaves on the log prismatic stack. We show that quasicoherent sheaves on $(X, \mathcal{M})^\Delta$ recover log prismatic crystals. This ends up being slightly more subtle than the argument for X^Δ , because we must generally avoid using Δ_R for R quasiregular semiperfect. The basic issue is that in the log setting these will generally not come with fine log structures, so they will not admit maps to Log .

We first show strict log prisms give points of Olsson's log prismatic stack.

Lemma A.4. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$, and let $(A, I, \mathcal{N})^a$ be a bounded log prism with fine log structure and a strict map $(\mathrm{Spf} A/I, \mathcal{N}|_{\mathrm{Spf} A/I}) \rightarrow (X, \mathcal{M})$. Then there is a natural map*

$$\rho_{X,A} : \mathrm{Spf} A \rightarrow (X, \mathcal{M})^\Delta.$$

If $(X, \mathcal{M})$ is a framed log affine and $(A, I, \mathcal{N})^a$ is the logarithmic Breuil-Kisin prism associated with its framing, then $\rho_{X,A}$ is an fpqc epimorphism.

Proof. Construction 8.14 of [BL22b] gives a map $\mathrm{Spf} A \rightarrow X^\Delta$, using the prism structure. Using the definition of $(X, \mathcal{M})^\Delta$ as a relative prismatic stack with respect to Log , it remains to complete the diagram

$$\begin{array}{ccc} \mathrm{Spf} A & \longrightarrow & X^\Delta \\ \downarrow & & \downarrow \\ \mathbb{Z}_p^\Delta \times \mathrm{Log} & \xrightarrow{\pi_{\mathrm{Log}}} & \mathrm{Log}^\Delta. \end{array}$$

Currently we are missing the dashed arrow as we have not used the log structure, but this is supplied because $\mathcal{M}_{\mathrm{Spf} A}$ gives a map to Log , with the other component being the classifying map to $\mathbb{Z}_p^\Delta$; unwinding definitions and using that we have a log prism one may check the composites to Log^Δ agree, so we get the desired map.

Now suppose $(X, \mathcal{M})$ is a framed log affine. In this situation, Lemma A.21 supplies an fpqc epimorphism

$$\rho_{X,B_P} : \mathrm{Spf} B_P \rightarrow (X, \mathcal{M})^\Delta.$$

We may check whether $\rho_{X,A} : \mathrm{Spf} A \rightarrow (X, \mathcal{M})^\Delta$ is an fpqc epimorphism after base change along one, and indeed we obtain

$$\begin{array}{ccc} \mathrm{Spf}(A \coprod_* B_P) & \longrightarrow & \mathrm{Spf} B_P \\ \downarrow & & \downarrow \\ \mathrm{Spf} A & \xrightarrow{\rho_{X,A}} & (X, \mathcal{M})^\Delta \end{array}$$

using Lemma A.6 (which only uses the first part of this lemma). Here $(A \coprod_* B_P)$ is the coproduct in the category of log prisms $(X, \mathcal{M})_\Delta^{\mathrm{op}}$ which exists by Lemma A.5. But one can directly check that

$$\mathrm{Spf}(A \coprod_* B_P) \longrightarrow \mathrm{Spf} B_P$$

is fpqc as it is a prismatic envelope of a (p, I) -completely regular sequence, using [BS22, Proposition 3.13] (as constructed in Lemma A.5). This proves the claim. $\square$

Using these fpqc epimorphisms, we next want to relate fiber products $\mathrm{Spf} A \times_{(X, \mathcal{M})^\Delta} \mathrm{Spf} B$ to coproducts in the category of log prisms over $(X, \mathcal{M})$. However since [Kos22, Proposition 4.13] works with the relative rather than absolute site there is not a direct reference. We use a similar method to construct the envelope in our particular case.

Lemma A.5. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$ with $X = \mathrm{Spf} R$, let $(A, I, M_{\mathrm{Spf} A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing, and let $(B, (d), \mathcal{N})^a$ be an orientable bounded strict object of $(X, \mathcal{M})_\Delta$. The coproduct $A \coprod_* B$ in $(X, \mathcal{M})_\Delta^{\mathrm{op}}$ exists.*

Proof. We recall that the construction in Lemma A.3 has several variants: the first uses the usual Breuil-Kisin prism with prismatic ideal $E(u)$ generated by an Eisenstein polynomial of a uniformizer π , or we may use a log q -de Rham prism. The argument in each case is very similar, so we will focus on the first case (the answer in the q -de Rham case, at least for a self-coproduct, is already recorded in Lemma 3.6). The general argument is analogous to [Kos22, Proposition 4.13].

Write $R = R_0 \widehat{\otimes}_{W(k)} \mathcal{O}_K$, with the strict étale framing

$$\square : (\mathrm{Spf} R_0, \mathbf{N}^r) \longrightarrow (\mathrm{Spf} W(k) \langle T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1} \rangle, \mathbf{N}^r).$$

We keep $E(u)$ as a choice of Eisenstein polynomial for a choice of uniformizer π . From Lemma A.3, we construct $A = R_0[[u]]$ and $I = (E(u))$, where u, T_i are sent to 0 by δ ; it is equipped with a log structure with prelog chart $\mathbf{N}^r$ lifted along the framing.

We wish to construct the coproduct $A \coprod_* B$. For the other log prism $(B, (d), \mathcal{N})^a$, put $\bar{B} = B/d$. By strictness and [Kos22, Lemma 3.8] we get a chart $\mathbf{N}^r \rightarrow \Gamma(\mathrm{Spf} B, \mathcal{N})$ with $e_i \mapsto m_i \in \Gamma(\mathrm{Spf} B, \mathcal{N})$ (using the lemma to lift the sections on $\Gamma(\mathrm{Spf} B/d, \mathcal{N}|_{B/d})$ which is pulled back from $\mathcal{M}$ by strictness). Write $\alpha : \Gamma(\mathrm{Spf} B, \mathcal{N}) \rightarrow B$ for the log structure map on global sections. For A , we denote by e_i the analogous log sections $\mathbf{N}^r \rightarrow A$ which are sent modulo $E(u)$ to T_i .

The coproduct $C = A \coprod_* B$ is characterized by its universal property, which we will match directly with an explicit construction. To do this we must construct a log prism C with maps from A and

B such that, for every bounded test log prism S over $(X, \mathcal{M})$,

$$\mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{C}, S) \simeq \mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{A}, S) \times \mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{B}, S).$$

For a pair $f : \mathcal{A} \rightarrow S$ and $g : \mathcal{B} \rightarrow S$ on the right side, rigidity of prisms shows the prismatic ideal of S is dS . We work with the strict site and f and g are morphisms of prisms over $(X, \mathcal{M})$, so log sections $g(m_i)$ and $f(e_i)$ are forced to have the same reduction. Then Lemma 3.8 of [Kos22] gives us a unique unit $\lambda_i \in 1 + dS$ such that $g(m_i) = \lambda_i f(e_i)$.

Now we construct $\mathcal{C}$ with this property by matching with the universal property of a prismatic envelope. We adjoin variables v_i representing these units λ_i . Form

$$D_0^{\mathrm{ex}} = ((R_0[u] \otimes_{W(k)} B)[v_1^{\pm 1}, \dots, v_r^{\pm 1}] / (\alpha(m_i) - v_i T_i)_{i=1}^r)_{(p, d)}^{\wedge}.$$

Recall $\alpha : \Gamma(\mathrm{Spf} B, \mathcal{N}) \rightarrow B$ is the log structure map. Give this ring the log structure pulled back from B and set the δ (or rather $\delta_{\log}$) structure as $\delta(v_i) = v_i^p \delta_{\log}(m_i)$.¹⁶

Using the maps $R_0[u] \rightarrow R \rightarrow \overline{B}$, $B \rightarrow \overline{B}$, and $v_i \mapsto 1$ we obtain

$$q_0 : D_0^{\mathrm{ex}} \rightarrow \overline{B}.$$

Using the framing $\square$ and noting $T_i = \alpha(m_i)/v_i$ for $i \leq r$ to eliminate these variables we see D_0^{ex} is (p, d) -completely smooth over B of relative dimension $n + 1$ (these come from u , the remaining T_i for $r < i \leq n$, and the v_i). Locally $\ker(q_0) = (d, x_1, \dots, x_{n+1})$, with the x_j a (p, d) -completely regular sequence relative to B (in the sense of [BS22, Definition 2.42]). We finally construct $\mathcal{C}$ as the prismatic envelope

$$\mathcal{C} = (D_0^{\mathrm{ex}} \{x_1/d, \dots, x_{n+1}/d\}_{\delta})_{(p, d, E)}^{\wedge},$$

with the naturally induced log structure from D_0^{ex} . The map $B \rightarrow \mathcal{C}$ makes it so $\mathcal{C}$ is a strict log prism in $(X, \mathcal{M})_{\Delta}^{\mathrm{op}}$. Since $E(u) \in \ker(q_0)$ maps into $d\mathcal{C}$, the map $R_0[u] \rightarrow \mathcal{C}$ extends to a log prism map $\mathcal{A} \rightarrow \mathcal{C}$.

Now we need to match the universal properties. Recall that the universal property of this prismatic envelope is that for each prism (S, I_S) and δ -map

$$h : D_0^{\mathrm{ex}} \longrightarrow S$$

carrying $\ker(q_0)$ into I_S there is a unique compatible map $\mathcal{C} \rightarrow S$.

Now suppose we have for some test log prism S , two log prism maps over $(X, \mathcal{M})$: $\mathcal{A} \rightarrow S$ and $\mathcal{B} \rightarrow S$. Then the construction of D_0^{ex} makes it so that there is a unique $\delta_{\log}$ -map

$$D_0^{\mathrm{ex}} \longrightarrow S$$

that carries v_i to λ_i . The maps agree over $(X, \mathcal{M})$, forcing this map to also carry $\ker(q_0)$ into I_S . Now we use the universal property of the prismatic envelope: there is a unique compatible map $\mathcal{C} \rightarrow S$, which is forced to be a map of log prisms. This gives an injection

$$\mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{A}, S) \times \mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{B}, S) \hookrightarrow \mathrm{Hom}_{(X, \mathcal{M})_{\Delta}^{\mathrm{op}}}(\mathcal{C}, S).$$

It is surjective, since we can restrict a log prism map $h : \mathcal{C} \rightarrow S$ along the log prism maps $\mathcal{A} \rightarrow \mathcal{C}$ and $\mathcal{B} \rightarrow \mathcal{C}$ to obtain f and g . These must satisfy $g(m_i) = h(v_i)f(e_i)$ by construction, where $h(v_i) \in 1 + I_S$. But then by uniqueness we get $h(v_i) = \lambda_i \in 1 + I_S$, and it follows that the map

$$D_0^{\mathrm{ex}} \longrightarrow S$$

¹⁶This construction is realized as exactification of a $\delta_{\log}$ -triple as in [Kos22, Construction 2.17].

we then obtain is $h|_{D_0^{\text{ex}}}$, so by the previous procedure we recover h from f and g . $\square$

We now show how to recover these coproducts via the log prismatic stack.

Lemma A.6. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$ and let $(A, I, M_{\text{Spf } A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing. Let $(B, (d), \mathcal{N})^a$ be an bounded log prism in $(X, \mathcal{M})_{\Delta}^{\text{op}}$. Then*

$$\text{Spf}(A \coprod_{\ast} B) \simeq \text{Spf } A \times_{(X, \mathcal{M})^{\Delta}} \text{Spf } B.$$

The analogous assertion also holds with trivial log structures.

Proof. By Lemma A.5, the coproduct exists and therefore the statement makes sense. We will use the notation of Lemma A.5, with $\bar{B} = B/d$.

In the situation of a framed log affine, the log structure factors as $X \rightarrow \mathcal{A}_{N^r} \rightarrow \text{Log}$. Here $\mathcal{A}_{N^r} \rightarrow \text{Log}$ is an étale chart for the log stack, given by $\mathcal{A}_P := [\text{Spf } \mathbf{Z}_p \langle P \rangle / \text{Spf } \mathbf{Z}_p \langle P^{\text{gp}} \rangle]$. This fact is useful because in the definition of $(X, \mathcal{M})^{\Delta}$ it allows us to replace all instances of Log with $\mathcal{A}_{N^r}$. That is,

$$(X, \mathcal{M})^{\Delta} \simeq (X/\mathcal{A}_{N^r})^{\Delta}.$$

In what follows, we will use the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\text{id}} & X & \longleftarrow & \text{Spf } \bar{B} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spf } A & \longrightarrow & \mathcal{A}_{N^r} & \longleftarrow & \text{Spf } B. \end{array}$$

Taking relative prismatic stacks on the columns and noting $(X/\text{Spf } A)^{\Delta} \simeq \text{Spf } A$ and $(\text{Spf } \bar{B}/\text{Spf } B)^{\Delta} \simeq \text{Spf } B$, we get maps

$$\text{Spf } A \longrightarrow (X, \mathcal{M})^{\Delta} \longleftarrow \text{Spf } B.$$

Revisiting the construction in Lemma A.4, these are precisely $\rho_{X,A}$ and $\rho_{X,B}$.

Even better, using finite limit compatibility of the relative prismatic stack in this diagram (for δ -pairs by [AKN26, Lemma 4.20]) we obtain

$$\text{Spf } A \times_{(X, \mathcal{M})^{\Delta}} \text{Spf } B \simeq (\text{Spf } \bar{B} / (\text{Spf } A \times_{\mathcal{A}_{N^r}}^R \text{Spf } B))^{\Delta}.$$

We will compute the relative prismatic stack on the right, showing that it is discrete. Using the presentation $\mathcal{A}_{N^r} = [\text{Spf } \mathbf{Z}_p \langle \mathbf{N}^r \rangle / \mathbf{G}_m^r]$ we may compute the denominator $\text{Spf } A \times_{\mathcal{A}_{N^r}}^R \text{Spf } B$ explicitly as

$$\text{Spf} \left((A \otimes_{\mathbf{Z}_p}^L B)[v_1^{\pm 1}, \dots, v_r^{\pm 1}] // (\alpha(m_i) - v_i T_i)_{i=1}^r \right)_{(p, E, d)}^{\wedge}.$$

Here v_i are the coordinates on $\mathbf{G}_m^r$, and the map from $\text{Spf } \bar{B}$ is given by $v_i \mapsto 1$.

This is very similar to the construction in Lemma A.5, but with one difference. Let us recall exactly what this construction did. From Lemma A.5, the E-completion D^{ex} has the presentation

$$D^{\text{ex}} = ((A \otimes_{W(k)}^L B)[v_1^{\pm 1}, \dots, v_r^{\pm 1}] // (\alpha(m_i) - v_i T_i)_{i=1}^r)_{(p, E, d)}^{\wedge}.$$

Note this completion is harmless as ultimately we will get the same prismatic envelope. The difference is then clear: we have changed from $\mathbf{Z}_p$ to $W(k)$. The maps $W(k) \rightarrow \bar{B}$ induced by A and B agree. Now since $W(k)$ is p -completely formally étale over $\mathbf{Z}_p$, the canonical map

$$(\bar{B}/D^{\text{ex}})^{\Delta} \xrightarrow{\sim} (\text{Spf } \bar{B}/(\text{Spf } A \times_{\mathcal{A}_{N^r}}^R \text{Spf } B))^{\Delta}$$

is an equivalence (see [AKN26, Lemma 5.9]. The proof shows a comparison of stacks, even though it is stated for cohomology). Thus this distinction is irrelevant.

Now it remains to compute $(\bar{B}/D^{\text{ex}})^{\Delta}$. By [BL22b, Theorem 7.17] to do so, we then see

$$\text{Spf } A \times_{(X, \mathcal{M})^{\Delta}} \text{Spf } B \simeq \text{Spf } \Delta_{\bar{B}/D^{\text{ex}}}.$$

This may be computed directly as a prismatic envelope as $\bar{B}$ is locally cut out by a Koszul regular sequence in D^{ex}/d , and as previously observed this prismatic envelope is the same as the one computed using D_0^{ex} . Thus in Lemma A.5 we have shown this is $\text{Spf}(A \coprod_* B)$ where the coproduct is in the category of prisms $(X, \mathcal{M})_{\Delta}^{\text{op}}$.

The ordinary assertion follows by a similar (far easier) argument. $\square$

Proposition A.7. *Let $(X, \mathcal{M}) = (\text{Spf } R, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$, and let*

$$\rho_{X,A} : \text{Spf } A \rightarrow (X, \mathcal{M})^{\Delta}$$

be the fpqc epimorphism of Lemma A.4 associated with the logarithmic Breuil-Kisin prism of its framing. Then the Čech nerve of $\text{Spf } A$ identifies with $\text{Spf } A^{(n)}$ where $A^{(n)}$ are the iterated self-coproducts of $(A, I, M_{\text{Spf } A})^a$ in $(X, \mathcal{M})_{\Delta}^{\text{op}}$. In particular

$$(X, \mathcal{M})^{\Delta} \simeq |\text{Spf } A^{(\bullet)}|$$

in formal stacks.

Proof. Lemma A.6, applied with $B = A$, identifies the first term of the Čech nerve of $\text{Spf } A$ over $(X, \mathcal{M})^{\Delta}$ with $\text{Spf}(A \coprod_* A)$. The higher terms are the formal spectra of the iterated coproducts $(A^{(1)})^{\widehat{\otimes}_{A^n}}$, for both the site-theoretic Čech nerve and the Čech nerve of $\rho_{X,A} : \text{Spf } A \rightarrow (X, \mathcal{M})^{\Delta}$. Using Lemma A.4, this is an fpqc epimorphism so the claim follows. $\square$

Proposition A.8. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$. There is a natural symmetric monoidal equivalence*

$$\iota_{(X, \mathcal{M})} : D((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} D_{\text{qc}}((X, \mathcal{M})^{\Delta}).$$

Proof. We first note that the assertion can be checked strict étale locally. On a framed log affine, using the logarithmic Breuil-Kisin prism $(A, I, M_{\text{Spf } A})^a$ associated with its framing yields

$$\lim_{[n] \in \Delta} D_{(p, I^{(n)})}(A^{(n)}) \simeq D((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta})$$

where $A^{(n)}$ is the logarithmic Čech nerve. The (analogous) proof of Corollary A.18 for ordinary crystals gives strict étale descent of the crystal side, which is well-known.

The same descent claim holds for the stack $(X, \mathcal{M})^\Delta$ by étale descent for X^Δ in X from [BL22b]. We may therefore assume $(X, \mathcal{M}) = (\mathrm{Spf} R, \mathcal{M})$ is a framed log affine. The logarithmic Breuil-Kisin prism A associated with its framing covers the log prismatic stack by Lemma A.4.

Now we can use Proposition A.7 to identify

$$D_{\mathrm{qc}}((X, \mathcal{M})^\Delta) \simeq \lim_{[n] \in \Delta} D_{\mathrm{qc}}(\mathrm{Spf} A^{(n)}) \simeq D((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta).$$

Indeed, $\rho_{X,A} : \mathrm{Spf} A \rightarrow (X, \mathcal{M})^\Delta$ is an fpqc epimorphism and we have identified its Čech nerve with $\mathrm{Spf} A^{(n)}$, and thus D_{qc} satisfies descent for such maps. See the discussion after [Pen25, Definition 2.4] for more details. $\square$

For a strict morphism f , we write $Rf_{\Delta,*}$ and Rf_* for the right adjoints to pullback on the two categories of Proposition A.8, respectively.

Proposition A.9. *Let $f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ be a strict morphism between pairs satisfying Assumption 3.1 over $\mathcal{O}_K$. For every $\mathcal{E} \in D((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta)$, there is a natural equivalence*

$$\iota_{(Y, \mathcal{N})} Rf_{\Delta,*} \mathcal{E} \simeq Rf_* \iota_{(X, \mathcal{M})} \mathcal{E}.$$

Proof. Compatibility of Proposition A.8 with strict pullback gives a commutative diagram

$$\begin{array}{ccc} D((Y, \mathcal{N})_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\iota_{(Y, \mathcal{N})}} & D_{\mathrm{qc}}((Y, \mathcal{N})^\Delta) \\ \downarrow f^* & & \downarrow (f^\Delta)^* \\ D((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\iota_{(X, \mathcal{M})}} & D_{\mathrm{qc}}((X, \mathcal{M})^\Delta). \end{array}$$

Both vertical pullback functors preserve colimits, thus taking right adjoints of this square gives the asserted equivalence. $\square$

Remark A.10. Laurent F-crystals in [KY25, IKY26] use the saturated log prismatic site. By [Ino25, Lemma 5.9], the strict subsite is cofinal in the full log prismatic site. If $(X, \mathcal{M})$ is fine and saturated, a strict object has the same characteristic monoid as the pullback of $\overline{\mathcal{M}}$, and is therefore saturated. The strictification in [Ino25, Lemma 5.9] has this same characteristic monoid, so the strict subsite is also cofinal in the saturated log prismatic site. Thus the limits defining the crystal categories on the full, saturated, and strict sites agree.

We suppress this distinction for $D((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta)$ and its perfect and Laurent variants for this reason.

A.3. Solid Laurent crystals. In this subsection, we define a solid generalization of

$$\mathrm{Perf}((X, \mathcal{M})_\Delta, \mathcal{O}_\Delta[1/I_\Delta]_p^\wedge),$$

and show that in nice situations we can still understand it explicitly through Čech nerves.

Definition A.11. Let $(A, (d))$ be an orientable prism. Associated to the derived (p, d) -complete ring A is the analytic adic ring (A, A) in the sense of [AM24]. We define

$$D_{\blacksquare}(A[1/d]_p^{\wedge}) := \varprojlim_n \text{Mod}_{(A//p^n)[1/d]} D_{\blacksquare}((A, A)),$$

where $(A//p^n)[1/d]$ is regarded as an algebra object of $D_{\blacksquare}((A, A))$. Equivalently, this is $D_{\blacksquare}$ of $\text{colim}_n \text{Spa}((A//p^n)[1/d], A//p^n)$, where we use Spa to denote the analytic stack attached to a Huber pair as in [Kip26a, Example 2.1.5].

Then

$$D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) := \lim_{(A, (d), M_{\text{Spf } A})^a \in (X, \mathcal{M})_{\Delta}^{\text{op}}} D_{\blacksquare}(A[1/d]_p^{\wedge}).$$

The definition is independent of the generator d , and one may locally assume that the prismatic ideal is principal.

In what follows we repeatedly use the following formal fact: if $\mathcal{C} \simeq \varprojlim_i \mathcal{C}_i$ is a limit of presentable symmetric monoidal stable ∞ -categories and an algebra $S \in \mathcal{C}$ has images S_i , then

$$\text{Mod}_S(\mathcal{C}) \simeq \varprojlim_i \text{Mod}_{S_i}(\mathcal{C}_i).$$

Indeed, algebra objects and their module objects are computed in limits. We define perfect Laurent crystals by dualizability:

$$\text{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) := D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})^{\text{dual}}.$$

Our first task is to show that $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ can be computed using a Čech nerve. This will require us to prove some descendability results for log prismatic envelopes. The following criterion will be useful for this.

Lemma A.12. *Let $A \rightarrow B$ be a faithfully flat map of connective animated rings. If $\pi_0 B$ is countably presented as a $\pi_0 A$ -module, then $A \rightarrow B$ is descendable of index at most 2.*

Proof. See [Mat16, Corollary 3.33 and the proof of Proposition 3.32]. □

Lemma A.13. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$, let $(A, I, M_{\text{Spf } A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing, and let $(B, (d), \mathcal{N})^a \in (X, \mathcal{M})_{\Delta}^{\text{op}}$. Put*

$$C = A \coprod_{*} B$$

for the coproduct in $(X, \mathcal{M})_{\Delta}^{\text{op}}$. Then

$$B \longrightarrow C$$

is completely descendable for the (p, d) -adic topologies in the sense of [Kip26b, Definition 2.4.1.7], and the induced algebra is descendable in $D_{\blacksquare}((B, B))$.

Moreover, $\text{Spa}(C, C) \rightarrow \text{Spa}(B, B)$ is a proper descendable cover.

Note that as a degenerate case the same assertions hold when X is smooth over $\mathcal{O}_K$, (A, I) is an ordinary Breuil-Kisin atlas obtained from a smooth lift over $W(k)[[u]]$, $(B, (d)) \in X_{\Delta}$, and $C = A \coprod_{*} B$.

Proof. We prove the logarithmic assertion; the ordinary case is identical.

By the construction in Lemma A.5, Zariski locally we have

$$(4) \quad C \simeq (D^{\text{ex}}\{x_1/d, \dots, x_m/d\}_\delta)_{(p,d)}^\wedge,$$

where $\ker(D^{\text{ex}} \rightarrow \bar{B}) = (d, x_1, \dots, x_m)$. Using [Mao24, Proposition 5.52, and Remark 5.54] we see that $B \rightarrow C$ is (p, d) -completely faithfully flat. Its derived reductions modulo (p^k, d^k) are faithfully flat and countably presented on π_0 as modules, since we adjoin finitely many variables as a δ ring. By Lemma A.12, these reductions are then descendable of index ≤ 2 .

However for our application, we need complete descendability, as in [Kip26b, Definition 2.4.1.7]. Using rigidity of prisms, we see $B \rightarrow C$ is adic. Put $J = \text{fib}(B \rightarrow C)$ in the derived (p, d) -complete category and write $J_k = J // (p^k, d^k)$. Derived completeness then gives

$$\text{RHom}_B(J \widehat{\otimes}_B J, B) \simeq \text{R} \lim_k \text{RHom}_{B // (p^k, d^k)}(J_k^{\otimes 2}, B // (p^k, d^k)).$$

As we have seen already, the canonical map $J \widehat{\otimes}_B J \rightarrow B$ is nullhomotopic modulo (p^k, d^k) . One may then still apply the argument of [BS17, Lemma 11.22] (despite that it applies to $\mathbf{N}$ -indexed colimits) to show that $(J \widehat{\otimes}_B J)^{\otimes 2} \rightarrow B$ is nullhomotopic and therefore that C is descendable of index ≤ 4 in the derived (p, d) -complete category as desired.

Under $D(\text{Spf } B) \simeq D_{(p,d)\text{-comp}}(B)$, the analytification functor used in the proof of [Kip26a, Theorem 2.3.3(A)] preserves descendable algebras. Hence the algebra induced by C in $D_\bullet((B, B))$ is descendable as desired.

It remains to prove properness. Example 2.1.13 of [Kip26a] says that a map $\text{Spa}(C, C) \rightarrow \text{Spa}(B, B)$ is proper if and only if C is integral over the subring generated by B and topologically nilpotent elements of C . In the present situation, it is therefore enough to prove that

$$\pi_0(B // (p, d)) \longrightarrow \pi_0(C // (p, d))$$

is integral. Indeed, if a monic polynomial f over $\pi_0 B$ annihilates c modulo (p, d) , then $f(t) - f(c)$ is a monic polynomial over $\pi_0 B + (p, d)\pi_0 C$ annihilating c .

Applying Lemma A.14 to (4), using $D^{\text{ex}}/(d, x_1, \dots, x_m) \simeq B/d$, gives the needed integrality and hence properness (as for maps of Huber pairs regarded as analytic rings, the property of being affine proper is detected when the map on $+$ rings is integral up to topologically nilpotent elements). The same argument applies to the ordinary case. $\square$

Lemma A.14. *Let A be an animated δ -ring, let $d \in \pi_0 A$ be distinguished, and let $\{f_i\}$ be any (arbitrarily large) family of elements of $\pi_0 A$. Suppose A is further derived (p, d) complete. Then the morphism*

$$\text{Spf}(A\{f_i/d\}_\delta)_{(p,d)}^\wedge \longrightarrow \text{Spf } A$$

induced by the structure map is integral in the sense of [Kip26a, 2.2.5]. In other words, on rings it is true in the ordinary sense modulo (p, d) .

Proof. As we discuss integrality on formal schemes, we must show

$$\pi_0(A // (p, d)) \longrightarrow \pi_0\left((A\{f_i/d\}_\delta)_{(p,d)}^\wedge // (p, d)\right)$$

is integral. It suffices to check all of the iterates $\delta^n(f_i/d)$ satisfy monic polynomials in $\pi_0(A// (p, d))[T]$ modulo (p, d) . We will show that $\delta^n(f_i/d)$ satisfies a monic polynomial in $\pi_0(A// (p, d))[(f_i/d), \delta(f_i/d), \dots, \delta^{n-1}(f_i/d)]$.

The key identity for us will be

$$0 = \delta^n(d(f_i/d) - f_i) = \varphi^n(d)\delta^n(f_i/d) + F_n((f_i/d), \delta(f_i/d), \dots, \delta^{n-1}(f_i/d))$$

where F_n is a polynomial with coefficients in A . Moreover, for $n \geq 1$, modulo (p, d) , F_n has degree p in $\delta^{n-1}(f_i/d)$, with leading coefficient $\delta(\varphi^{n-1}(d))$. We will drop the argument of F_n from here on.

The identity holds for $n = 0$ with $F_0 = -f_i$. Applying δ to the relation, one sees that $\delta^{n+1}(d(f_i/d) - f_i)$ equals

$$\delta(\varphi^n(d)\delta^n(f_i/d)) + \delta(F_n) - \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} (\varphi^n(d)\delta^n(f_i/d))^j F_n^{p-j}.$$

One computes $\delta(\varphi^n(d)\delta^n(f_i/d)) = \varphi^{n+1}(d)\delta^{n+1}(f_i/d) + \delta(\varphi^n(d))(\delta^n(f_i/d))^p$ as well. Collecting the remaining terms to form F_{n+1} gives the first part of the claim. For the second part, if we work modulo (p, d) , the previous display becomes $\delta(\varphi^n(d)\delta^n(f_i/d)) + \delta(F_n)$, as the factor we subtract has all terms divisible by $\varphi^n(d) \in (p, d)$. Since F_n involves only the preceding iterates, $\delta(F_n)$ has degree at most one in $\delta^n(f_i/d)$ modulo p . Thus the degree p term of F_{n+1} has the asserted coefficient.

To finally deduce the desired claim for $\delta^n(f_i/d)$, one then reduces this relation for the index $n + 1$ modulo (p, d) : this kills the term $\varphi^{n+1}(d)\delta^{n+1}(f_i/d)$, leaving only the remaining terms $F_{n+1}((f_i/d), \delta(f_i/d), \dots, \delta^n(f_i/d))$. We know the leading coefficient in $\delta^n(f_i/d)$ for F_{n+1} modulo (p, d) was $\delta(\varphi^n(d))$ attached to $(\delta^n(f_i/d))^p$, but φ and δ commute so this is a unit (since d is distinguished; i.e. $\delta(d)$ is a unit). Thus after we divide by this unit, we obtain a monic polynomial in $\pi_0(A// (p, d))[(f_i/d), \delta(f_i/d), \dots, \delta^{n-1}(f_i/d)]$ that $\delta^n(f_i/d)$ satisfies modulo (p, d) . The claim follows. $\square$

Theorem A.15. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$, and let $(A, I, M_{\mathrm{Spf} A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing. If $(A^{(m)}, I, M^{(m)})^a$ is the degree m term of its Čech nerve in the strict log prismatic site, then*

$$D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \lim_{[m] \in \Delta} D_{\blacksquare}(A^{(m)}[1/I]_p^{\wedge}).$$

Proof. By definition,

$$D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \lim_{(B, (d), M_{\mathrm{Spf} B})^a \in (X, \mathcal{M})_{\Delta}^{\mathrm{op}}} D_{\blacksquare}(B[1/d]_p^{\wedge})$$

where $D_{\blacksquare}(B[1/d]_p^{\wedge})$ is interpreted as in Definition A.11. Now Lemma A.13 tells us

$$D_{\blacksquare}(B[1/d]_p^{\wedge}) \simeq \lim_{[m] \in \Delta} D_{\blacksquare}((A^{(m)} \coprod_{*} B)[1/d]_p^{\wedge}).$$

Now upon commuting limits, we get

$$D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \lim_{[m] \in \Delta} \left(\lim_{(B, (d), M_{\text{Spf } B})^a \in (X, \mathcal{M})_{\Delta}^{\text{op}}} D_{\blacksquare}((A^{(m)} \coprod_* B)[1/d]_p^{\wedge}) \right).$$

We are now done, as formally we have

$$\lim_{(B, (d), M_{\text{Spf } B})^a \in (X, \mathcal{M})_{\Delta}^{\text{op}}} D_{\blacksquare}((A^{(m)} \coprod_* B)[1/d]_p^{\wedge}) \simeq D_{\blacksquare}(A^{(m)}[1/I]_p^{\wedge}),$$

proving the claim. One way to see this last point is that the coproduct functor

$$(X, \mathcal{M})_{\Delta}^{\text{op}} \longrightarrow (X, \mathcal{M})_{\Delta, /A^{(m)}}^{\text{op}}$$

sending $(B, (d), M_{\text{Spf } B})^a \mapsto (A^{(m)} \coprod_* B)$ regarded as a log prism is left adjoint to the forgetful functor, so by adjunction the limit is

$$\lim_{A^{(m)} \rightarrow B} D_{\blacksquare}(B[1/d]_p^{\wedge}) \simeq D_{\blacksquare}(A^{(m)}[1/I]_p^{\wedge}).$$

Thus $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \lim_{[m] \in \Delta} D_{\blacksquare}(A^{(m)}[1/I]_p^{\wedge})$. $\square$

Lemma A.16. *Let $(A, (d))$ be an orientable transversal prism, i.e. A/d is p -torsionfree. Then we have*

$$D_{\blacksquare}(A[1/d]_p^{\wedge})^{\text{dual}} \simeq \varprojlim_n \text{Perf}((A/p^n)[1/d]) \simeq \text{Perf}(A[1/d]_p^{\wedge}).$$

In the setting of Theorem A.15, these give

$$\text{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \varprojlim_{[m] \in \Delta} \text{Perf}(A^{(m)}[1/I]_p^{\wedge}).$$

Proof. This follows by taking dualizable objects in Theorem A.15 and applying [Sch26b, Theorem 14.10], noting dualizables can be detected termwise in a limit diagram. $\square$

Lemma A.17. *Let $(\text{Spf } R, \mathcal{M})$ be a framed log affine over $\text{Spf } W(k)[\zeta_p]$. Let $((A^+)^{(m)}, [p]_q, \mathcal{M}^{(m)})^a$ be the degree m term of the log q -de Rham Čech nerve in $(\text{Spf } R, \mathcal{M})_{\Delta}$, and put $A^{(m)} = (A^+)^{(m)}[1/[p]_q]_p^{\wedge}$. Then*

$$D_{\blacksquare}((\text{Spf } R, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) \simeq \varprojlim_{[m] \in \Delta} D_{\blacksquare}(A^{(m)}).$$

Proof. In Lemma A.3, the log q -de Rham prism is a valid choice of a logarithmic Breuil-Kisin atlas. Theorem A.15 therefore gives the needed Čech description. $\square$

Corollary A.18. *On log formal schemes satisfying Assumption 3.1 over $\mathcal{O}_K$, the assignment*

$$(X, \mathcal{M}) \longmapsto D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$$

is a stack for the strict étale topology. The same holds for $\text{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$.

Proof. Framed log affines form a basis for the strict étale topology by Lemma 3.4, so it is enough to consider an affine strict étale cover $U \rightarrow X$ by framed log affines with X framed. We give every term $U^{(n)}$ of its Čech nerve the framing induced from X , and let A_X be the logarithmic Breuil-Kisin prism associated with the framing of X . The étale lifting property identifies its pullback to each $U^{(n)}$ with the logarithmic Breuil-Kisin prism $A_{U^{(n)}}$ associated with the induced framing. Writing $A_X^{(m)}$ and $A_{U^{(n)}}^{(m)}$ for the Laurent rings in degree m of their Čech nerves, they form an (augmented) bicosimplicial diagram as follows:

$$\begin{array}{ccccc}
D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) & \longrightarrow & D_{\blacksquare}(A_X^{(0)}) & \rightrightarrows & D_{\blacksquare}(A_X^{(1)}) \rightrightarrows \cdots \\
\downarrow & & \downarrow & & \downarrow \\
D_{\blacksquare}((U, \mathcal{M}|_U)_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) & \longrightarrow & D_{\blacksquare}(A_U^{(0)}) & \rightrightarrows & D_{\blacksquare}(A_U^{(1)}) \rightrightarrows \cdots \\
\Downarrow & & \Downarrow & & \Downarrow \\
D_{\blacksquare}((U^{(1)}, \mathcal{M}|_{U^{(1)}})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) & \longrightarrow & D_{\blacksquare}(A_{U^{(1)}}^{(0)}) & \rightrightarrows & D_{\blacksquare}(A_{U^{(1)}}^{(1)}) \rightrightarrows \cdots \\
\Downarrow & & \Downarrow & & \Downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

For fixed n , Theorem A.15 identifies the limit of the n th row with the solid Laurent crystal category of $U^{(n)}$. For fixed m , compatibility with strict étale localization also identifies the m th column with the Čech nerve of an étale cover. For such maps $D_{\blacksquare}(\mathrm{Spa}(A, A))$ satisfies descent by [Kip26a, Theorem 2.3.3(A),(B)], which suffices for our situation: since limits commute with limits, we obtain

$$\begin{aligned}
\varprojlim_{[n] \in \Delta} D_{\blacksquare}((U^{(n)}, \mathcal{M}|_{U^{(n)}})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}) &\simeq \varprojlim_{[n] \in \Delta} \varprojlim_{[m] \in \Delta} D_{\blacksquare}(A_{U^{(n)}}^{(m)}) \\
&\simeq \varprojlim_{[m] \in \Delta} \varprojlim_{[n] \in \Delta} D_{\blacksquare}(A_{U^{(n)}}^{(m)}) \\
&\simeq \varprojlim_{[m] \in \Delta} D_{\blacksquare}(A_X^{(m)}) \\
&\simeq D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}).
\end{aligned}$$

The claim follows for $\mathrm{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ since dualizability can be tested termwise in a limit of symmetric monoidal categories. $\square$

The canonical t -structure on $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ is defined by taking the connective objects or ≤ 0 part to be the crystals whose evaluations on every prism are connective, and taking the coconnective objects to be the right orthogonal of $D_{\blacksquare}^{\leq -1}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$.

Corollary A.19. *Let $(X, \mathcal{M})$ satisfy Assumption 3.1 over $\mathcal{O}_K$. The canonical t -structure on $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ restricts to $\mathrm{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$.*

Proof. Using a covering by framed log affines via Lemma 3.4, we may use their associated logarithmic Breuil-Kisin prisms $(A, I, M_{\mathrm{Spf} A})^a$ from Lemma A.3 to produce a covering family (by

Theorem A.15). The argument ends up being slightly technical since there is a subtlety that it is not clear the maps in the Breuil-Kisin Čech nerve are t -exact for $D_{\blacksquare}(-)$.

First we simply check that there exists a t -structure on $\mathrm{Perf}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ regarded as ordinary Laurent crystals, defined via orthogonality from the Breuil-Kisin presentation of $\mathrm{Perf}^{\leq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$: we ask for the connective objects to be those which are connective on each Čech term $A^{(n)}[1/I]_p^{\wedge}$. This follows by the abstract argument of [Bha22, Footnote 70], noting that we actually only need the maps $A[1/I]_p^{\wedge} \rightarrow A^{(n)}[1/I]_p^{\wedge}$ to be flat (which they are in the ordinary sense since $A[1/I]_p^{\wedge}$ is Noetherian, using that p -complete flatness upgrades to ordinary flatness with a Noetherian source). By Lemma A.16 this also yields a t -structure on the dualizable objects in $D_{\blacksquare}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ as dualizability is detected termwise in a limit.

One may check that any $\mathcal{E}$ in $\mathrm{Perf}^{\leq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ is also in $D_{\blacksquare}^{\leq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. We must check all evaluations on log prisms are connective, as we define the canonical t -structure on $D_{\blacksquare}^{\leq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ this way by orthogonality. By construction of $\mathrm{Perf}^{\leq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ we see the evaluation on $(A \coprod_{\ast} B)[1/I]^{\wedge}$ is connective (as it is obtained by pullback from the value on $(A, I, M_{\mathrm{Spf} A})^a$, which by definition is connective). In Lemma A.13 we argued $(A \coprod_{\ast} B)[1/I]^{\wedge}$ is a descendable algebra in $D_{\blacksquare}(B[1/J]_p^{\wedge})$. In particular, the base change functor $B[1/J]_p^{\wedge} \rightarrow (A \coprod_{\ast} B)[1/I]^{\wedge}$ is conservative: in this situation base change detects connectivity on perfect complexes.

One may also check that $\mathcal{E} \in \mathrm{Perf}^{\geq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ is also in $D_{\blacksquare}^{\geq 0}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$ for its canonical t -structure. Indeed this is defined as the right orthogonal of

$$D_{\blacksquare}^{\leq -1}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge}).$$

For $\mathcal{E}' \in D_{\blacksquare}^{\leq -1}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$, every evaluation on Čech terms $A^{(n)}[1/I]_p^{\wedge}$ lies in the ≤ -1 part of the solid t -structure. Writing $\mathcal{E}_n$ and $\mathcal{E}'_n$ for these evaluations, we get

$$\mathrm{Map}_{D_{\blacksquare}(A^{(n)}[1/I]_p^{\wedge})}(\mathcal{E}'_n, \mathcal{E}_n) = 0$$

using that $\mathcal{E}_n$ are in the ≥ 0 part (using that it is perfect here to relate ordinary and solid t -structures). But mapping spaces are computed by the limit over the Čech nerve, so the claim of the corollary follows. $\square$

We denote its heart by $\mathrm{Coh}((X, \mathcal{M})_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]_p^{\wedge})$. Corollary A.18 then gives strict étale descent for its heart.

A.4. Analytifications. We show in this subsection that the Breuil-Kisin covers give proper effective epimorphisms after analytification, and use this to deduce presentations of $X^{\Delta, \blacksquare}$ and $(X, \mathcal{M})^{\Delta, \blacksquare}$ similar to what we had for the ordinary prismatic and log prismatic stacks.

Proposition A.20. *Let X be an affine smooth p -adic formal scheme over $\mathcal{O}_K$, and let (A, I) be an ordinary Breuil-Kisin atlas obtained from a smooth lift over $W(k)[[u]]$. The induced map*

$$\mathrm{Spa}(A, A) \longrightarrow X^{\Delta, \blacksquare}$$

is a representable proper effective epimorphism. If $A^{(n)}$ denotes the n th iterated self-coproduct in the category of prisms over X , then

$$X^{\Delta, \blacksquare} \simeq |\mathrm{Spa}(A^{(\bullet)}, A^{(\bullet)})|$$

in analytic stacks.

Proof. Choose a classical regular semiperfectoid cover of X in the infinite-root/étale topology. Construction 3.7.1 and Corollary 3.6.3 of [Kip26a] show that

$$\coprod_i \mathrm{Spa}(B_i, B_i) \longrightarrow X^{\Delta, \blacksquare}$$

is an effective epimorphism; after a Zariski refinement, the B_i are orientable bounded prisms over X . It therefore suffices to test the first assertion after pullback to each $\mathrm{Spa}(B_i, B_i)$. Lemma A.6 and finite limit preservation of analytic realization $(-)^{\blacksquare}$ (Definition 6.1) identify this pullback with

$$\mathrm{Spa}(A \coprod_* B_i, A \coprod_* B_i) \longrightarrow \mathrm{Spa}(B_i, B_i),$$

which is a proper descendable cover by Lemma A.13. Representability is local on the target, as is properness by [Kip26a, Remark 2.3.1]; effective epimorphisms are local on the target in any ∞ -topos. This proves the first assertion. Applying Lemma A.6 iteratively with $B = A$ identifies the Čech terms with $\mathrm{Spa}(A^{(n)}, A^{(n)})$. The resulting Čech nerve realizes to $X^{\Delta, \blacksquare}$ because $\mathrm{Spa}(A, A) \rightarrow X^{\Delta, \blacksquare}$ is an effective epimorphism. $\square$

The logarithmic version of this proposition holds, but it is far less obvious because we first have to produce a single effective epimorphism from a log prism to carry out a similar strategy. For a fine monoid P , let $\mathcal{A}_P = [\mathrm{Spf} \mathbf{Z}_p \langle P \rangle / \mathrm{Spf} \mathbf{Z}_p \langle P^{\mathrm{gp}} \rangle]$ be its Artin cone. This comes equipped with a natural cover $\mathrm{Spf} \mathbf{Z}_p \langle P \rangle \rightarrow \mathcal{A}_P$, and we have an étale map $\mathcal{A}_P \rightarrow \mathrm{Log}$ by [Ols03, Corollary 5.24].

The purpose of this first lemma is to produce the replacement for the semiperfectoid effective epimorphism.

Lemma A.21. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$, let $(A, I, M_{\mathrm{Spf} A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing, and let $P = \mathbf{N}^r$ be its framing's chart. There is an orientable bounded fine log prism $(B_P, I, \mathcal{N}_P)^a$ with a strict map*

$$(\mathrm{Spf}(B_P/I), \mathcal{N}_P|_{\mathrm{Spf}(B_P/I)}) \longrightarrow (X, \mathcal{M})$$

yielding an effective epimorphism $\mathrm{Spa}(B_P, B_P) \longrightarrow (X, \mathcal{M})^{\Delta, \blacksquare}$. In formal stacks the map

$$\mathrm{Spf} B_P \longrightarrow (X, \mathcal{M})^{\Delta}$$

is also an effective epimorphism.

Proof. Since $P = \mathbf{N}^r$ anyway, we will just write $\mathbf{N}^r$ directly in the proof. We will first construct the effective epimorphism

$$\mathrm{Spf} B_{\mathbf{N}^r} \longrightarrow (X, \mathcal{M})^{\Delta}$$

of fpqc stacks, and argue that after applying the functor $(-)^{\blacksquare}$ from Definition 6.1 we obtain an effective epimorphism of analytic stacks.

Since we work with a framed log affine, when constructing $(X, \mathcal{M})^{\Delta}$ we can realize it as the relative stack $(X/\mathcal{A}_{\mathbf{N}^r})^{\Delta}$. Writing the stack this way reveals that there is an fpqc epimorphism

$$\mathrm{Spf} A \times_{\mathcal{A}_{\mathbf{N}^r}^{\Delta}} (\mathbf{Z}_p^{\Delta} \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle) \longrightarrow (X, \mathcal{M})^{\Delta}.$$

Indeed, termwise $\mathrm{Spf} A \rightarrow X^{\Delta}$ is an fpqc epimorphism, as is $\mathbf{Z}_p^{\Delta} \times \mathrm{Spf} \mathbf{Z}_p \langle P \rangle \rightarrow \mathbf{Z}_p^{\Delta} \times \mathcal{A}_{\mathbf{N}^r}$. Thus we must construct an fpqc epimorphism

$$\mathrm{Spf} B_P \rightarrow \mathrm{Spf} A \times_{\mathcal{A}_{\mathbf{N}^r}^{\Delta}} (\mathbf{Z}_p^{\Delta} \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle).$$

This follows from rewriting the target stack in a different way. The main issue is that $\mathcal{A}_{\mathbf{N}^r}$ is a stack – if it was a smooth p -adic formal scheme, we could try to relate the target to Spf of some coproduct of prisms, which would give the desired claim.

What fixes this is that $\mathcal{A}_{\mathbf{N}^r}$ has affine diagonal. Let us rewrite the target using the Cartesian square

$$\begin{array}{ccc} \mathrm{Spf} A \times_{\mathcal{A}_{\mathbf{N}^r}^\Delta} (\mathbf{Z}_p^\Delta \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle) & \longrightarrow & X^\Delta \times_{X^\Delta} \mathrm{Spf} A \\ \downarrow & & \downarrow \\ \mathbf{Z}_p \langle \mathbf{N}^r \rangle^\Delta \times_{\mathbf{Z}_p \langle \mathbf{N}^r \rangle^\Delta} \mathbf{Z}_p^\Delta \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle & \longrightarrow & \mathcal{A}_{\mathbf{N}^r}^\Delta \times_{\mathbf{Z}_p^\Delta} \mathbf{Z}_p^\Delta. \end{array}$$

The idea is to trivially rewrite each term, so that upon commuting limits with limits we get

$$(X \times \mathbf{G}_m^r)^\Delta \times_{(X \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle)^\Delta} \mathrm{Spf}(A \hat{\otimes} \mathbf{Z}_p \langle \mathbf{N}^r \rangle).$$

The key thing we have used here is that $(-)^{\Delta}$ is compatible with fiber products, and $\mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle \times_{\mathcal{A}_{\mathbf{N}^r}} X$ becomes affine. For the proof, in the upper right corner we could have equally well chosen $\mathbf{Z}_p \langle \mathbf{N}^r \rangle$ again, and the entire argument still works. We only make this choice to reuse the Breuil-Kisin prism A in a moment.

This new presentation is substantially more useful, because there is a Breuil-Kisin prism $A \langle u_1^{\pm 1}, \dots, u_r^{\pm 1} \rangle_{(p, I)}$ with $\delta(u_i) = 0$ giving an fpqc epimorphism

$$\mathrm{Spf} A \langle u_1^{\pm 1}, \dots, u_r^{\pm 1} \rangle_{(p, I)} \longrightarrow (X \times \mathbf{G}_m^r)^\Delta.$$

Upon base change, we obtain an fpqc epimorphism

$$\mathrm{Spf} A \langle u_1^{\pm 1}, \dots, u_r^{\pm 1} \rangle_{(p, I)} \times_{(X \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle)^\Delta} \mathrm{Spf} A(\hat{\otimes} \mathbf{Z}_p \langle \mathbf{N}^r \rangle) \longrightarrow \mathrm{Spf} A \times_{\mathcal{A}_{\mathbf{N}^r}^\Delta} (\mathbf{Z}_p^\Delta \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle).$$

The source is now a fiber product of two prisms with their natural maps to $(X \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle)^\Delta$, exactly the (non-logarithmic) situation understood in Lemma A.6. Letting the coproduct of these prisms be $B_{\mathbf{N}^r}$ we obtain a composite fpqc epimorphism

$$\mathrm{Spf} B_{\mathbf{N}^r} \longrightarrow \mathrm{Spf} A \times_{\mathcal{A}_{\mathbf{N}^r}^\Delta} (\mathbf{Z}_p^\Delta \times \mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle) \longrightarrow (X, \mathcal{M})^\Delta.$$

This gives the second claim. Note that $B_{\mathbf{N}^r}$ is also a log prism as it naturally comes with a projection to $\mathbf{Z}_p^\Delta \times \mathcal{A}_P$ compatible with the map to $\mathcal{A}_P^\Delta$ using this map.

For the variant on analytic stacks, we apply $(-)^{\blacksquare}$ to both maps. The important property is that it is compatible with finite limits, so in all of the fiber products above we may simply apply it individually to each formal stack. For the first arrow, it is a proper effective epimorphism of analytic stacks by Proposition A.20.

For the second arrow, on the first factor we again apply Proposition A.20, and for the second factor we only use that

$$\mathrm{Spf} \mathbf{Z}_p \langle \mathbf{N}^r \rangle \longrightarrow \mathcal{A}_{\mathbf{N}^r}$$

is sent to an effective epimorphism of analytic stacks by $(-)^{\blacksquare}$ which is clear. $\square$

Proposition A.22. *Let $(X, \mathcal{M})$ be a framed log affine over $\mathcal{O}_K$, and let $(A, I, M_{\mathrm{Spf} A})^a$ be the logarithmic Breuil-Kisin prism associated with its framing. The induced map*

$$\mathrm{Spa}(A, A) \longrightarrow (X, \mathcal{M})^{\Delta, \blacksquare}$$

is a representable proper effective epimorphism. If $A^{(n)}$ denotes the n th iterated self-coproduct in $(X, \mathcal{M})_{\Delta}^{\text{op}}$, then

$$(X, \mathcal{M})^{\Delta, \blacksquare} \simeq |\text{Spa}(A^{(\bullet)}, A^{(\bullet)})|$$

in analytic stacks.

Proof. Let $P = \mathbb{N}^r$ be the framing's chart. Using Lemma A.21 we get an effective epimorphism

$$\text{Spa}(B_P, B_P) \longrightarrow (X, \mathcal{M})^{\Delta, \blacksquare}$$

induced by the map ρ_{X, B_P} induced by the log prism structure on B_P .

Lemma A.6 identifies the pullback of $\text{Spa}(A, A)$ along this map with

$$\text{Spa}(A \coprod_{*} B_P, A \coprod_{*} B_P) \longrightarrow \text{Spa}(B_P, B_P).$$

This is a proper descendable cover by Lemma A.13. Representability is local on the target, as is properness by [Kip26a, Remark 2.3.1]; the covering property is local in any ∞ -topos. This proves the first assertion.

Lemma A.6 and finite limit preservation of $(-)^{\blacksquare}$ (as in Definition 6.1) identify the n th Čech term with $\text{Spa}(A^{(n)}, A^{(n)})$. Since the map from $\text{Spa}(A, A)$ is an effective epimorphism, we deduce $(X, \mathcal{M})^{\Delta, \blacksquare} \simeq |\text{Spa}(A^{(\bullet)}, A^{(\bullet)})|$. $\square$

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