VECTOR BUNDLES ON THE FARGUES-FONTAINE CURVE

1. The main theorem

In this talk, I will be focusing on the classification of vector bundles on the Fargues-Fontaine curve following the exposition in Fargues-Scholze.

First, we will need to construct some of the relevant vector bundles. Throughout, let E be a finite extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_q$, ring of integers $\mathcal{O}_{\mathrm{E}}$ and a choice of uniformizer π. We will also put C as an algebraically closed perfectoid field over $\mathbf{F}_q$, and denote $\mathrm{X}_{C,\mathrm{E}}$ as X_C as E is implicit.

As a means for constructing vector bundles, we will use the category of isocrystals.

Definition 1.1. Let $\mathrm{E}/\mathbf{Q}_p$, and put $\breve{\mathrm{E}} = \mathrm{W}_{\mathcal{O}_{\mathrm{E}}}(\overline{\mathbf{F}}_q)[1/\pi]$ for the maximal unramified extension.

The category $\mathsf{Isoc}_{\mathrm{E}}$ is the E-linear $\otimes$-category with objects (V, φ) where $\mathrm{V} \in \mathsf{Vect}_{\breve{\mathrm{E}}}$ and $\varphi : \mathrm{V} \simeq \mathrm{V}$ is a $\varphi_{\breve{\mathrm{E}}}$-semilinear isomorphism.

For $\lambda = m/n \in \mathbf{Q}$ for m, n coprime and $n > 0$, we set V_λ to be the isocrystal with vector space $\breve{\mathrm{E}}^n$ and semilinear automorphism

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \pi^m & & & 0 \end{pmatrix} \varphi_{\breve{\mathrm{E}}}.$$

There is a functor

$$\mathsf{Isoc}_{\mathrm{E}} \to \mathsf{Vect}(\mathrm{X}_C)$$

arising from the observation that

$$\mathrm{Y}_{C,\mathrm{E}} \to \operatorname{Spa} \breve{\mathrm{E}}$$

and the structure morphism is equivariant for φ_C acting on $\mathrm{Y}_{C,\mathrm{E}}$ and $\varphi_{\breve{\mathrm{E}}}$ acting on $\operatorname{Spa} \breve{\mathrm{E}}$. Indeed, this induces a pullback functor

$$\mathsf{Isoc}_{\mathrm{E}} = \mathsf{Vect}^{\varphi_{\breve{\mathrm{E}}}}(\operatorname{Spa} \breve{\mathrm{E}}) \to \mathsf{Vect}^{\varphi_C}(\mathrm{Y}_{C,\mathrm{E}}).$$

But then by descent the latter is just $\mathsf{Vect}(\mathrm{X}_C)$.

DEFINITION 1.2. We set $\mathcal{O}(\lambda)$ to be the image of $\mathrm{V}_{-\lambda}$ under this map, so that $\mathcal{O}(1)$ is ample.

We are now ready to state the main theorem.

THEOREM 1.3 (Main theorem). There is a decomposition

$$\mathcal{E} \simeq \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}(\lambda)^{n_\lambda}$$

for any vector bundle $\mathcal{E} \in \mathsf{Vect}(\mathrm{X}_C)$.

Recalling the functor

$$\mathsf{Isoc}_{\mathrm{E}} \to \mathsf{Vect}(\mathrm{X}_C)$$

sends $\mathrm{V}_{-\lambda} \mapsto \mathcal{O}(\lambda)$, the Dieudonné-Manin decomposition

$$\mathsf{Isoc}_{\mathrm{E}} \simeq \bigoplus_{\lambda \in \mathbf{Q}} \mathsf{Isoc}_{\mathrm{E}}^{\lambda} = \bigoplus_{\lambda \in \mathbf{Q}} \mathrm{V}_\lambda \otimes \mathsf{Vect}_{\mathrm{E}}$$

implies this functor is a bijection on isomorphism classes.

REMARK 1.4. This generalizes to G-bundles and G-isocrystals. We can interpret a G-torsor on X_C as an exact $\otimes$-functor

$$\mathsf{Rep}_{\mathbf{Q}_p}(\mathrm{G}) \to \mathsf{Vect}(\mathrm{X}_C).$$

Then understanding $\mathsf{Vect}(\mathrm{X}_C)$ sufficiently well, i.e. the previous decomposition, produces a functor

$$\mathsf{Rep}_{\mathbf{Q}_p}(\mathrm{G}) \to \mathbf{Q} - \mathsf{FilVB}(\mathrm{X}_C)^{\mathrm{HN}},$$

the category of $\mathbf{Q}$-filtered vector bundles on X_C such that the $\lambda \in \mathbf{Q}$ component $\mathcal{E}^\lambda$ is semistable of slope λ. It's easy to check this is an exact $\otimes$-functor: exactness follows from $\mathsf{Rep}_{\mathbf{Q}_p}(\mathrm{G})$ being semisimple, and we can use the previous classification of vector bundles to check it is a $\otimes$-functor.

This allows us to produce an associated graded exact $\otimes$-functor

$$\mathsf{Rep}_{\mathbf{Q}_p}(\mathrm{G}) \to \mathsf{Isoc}_{\mathbf{Q}_p},$$

which is precisely the data of a G-isocrystal in $\mathrm{B}(\mathrm{G})$. To show this classifies the isomorphism classes of vector bundles we just need to split the previous filtration, which is done by computing $\mathrm{H}^1(\mathrm{X}_C, \mathcal{O}(\lambda)) = 0$ for $\lambda > 0$ so there are no extensions.

2. Ampleness of $\mathcal{O}(1)$

In the last section, we defined $\mathcal{O}(1)$ to be the image of the isocrystal V_{-1} under the functor

$$\mathsf{Isoc}_{E} \to \mathsf{Vect}(X_C).$$

It will be important for the argument to verify that $\mathcal{O}(1)$ is ample, or that $\mathcal{E}(n)$ is globally generated and $H^1(\mathcal{E}(n)) = 0$ for $n \gg 0$.

The reason we care about this is that it will give an injection

$$\mathcal{O}_{X_C}(-d) \to \mathcal{E}$$

for an arbitrary vector bundle. Indeed, a sufficiently large twist of $\mathcal{E}$ will then be globally generated and in particular admit a section, so upon untwisting we get the desired map.

> Theorem 2.1 (Kedlaya-Liu). Let $S/\mathbf{F}_q$ be an affinoid perfectoid space $\mathrm{Spa}(R, R^+)$, and let $\mathcal{E}$ be a vector bundle on $X_{S,E}$. Then there is some n_0 such that for all $n \geq n_0$ the vector bundle $\mathcal{E}(n)$ is globally generated and $H^1(X_{S,E}, \mathcal{E}(n)) = 0$.

Sketch. The proof is quite complicated and technical, so we will only give the basic idea of how to approach the question. We'll focus on showing H^1 vanishes.

Noting that the Frobenius φ_S multiplies the radius by q, so we can present

$$X_S = Y_S/\varphi^{\mathbf{Z}} = Y_{S,[1,q]}/(Y_{S,[1,1]} \sim Y_{S,[q,q]}).$$

Here, $Y_{S,I}$ is the open affinoid annulus $\mathrm{rad}^{-1}(I)$ for the radius function

$$\mathrm{rad} : |Y_S| \to (0, \infty).$$

Explicitly,

$$Y_{S,[a,b]} = \{|\pi|^b \leq |[\varpi]| \leq |\pi^a|\} \subset Y_S.$$

An immediate consequence of this presentation is that upon building a Čech complex computing cohomology, one obtains

$$R\Gamma(X_S, \mathcal{E}) = [\mathcal{E}(Y_{S,[1,q]}) \to \mathcal{E}(Y_{S,[q,q]})]$$

via $\varphi_S - 1$. By vanishing for affinoids, with no work we see H^2 vanishes. To get H^1 to vanish, you need to check this map is surjective for a sufficiently large twist.

The Čech approach allows us to reduce this to a commutative algebra question: any $\mathcal{E}$ can be written a finite projective $B_{R,[1,q]}$-module M with an isomorphism on its base changes

$$\varphi_M : M_{[q,q]} \simeq M_{[1,1]}$$

which is linear over φ.

Kedlaya-Liu show that one can reduce to the case where M is free, and in this case φ_M is given quite explicitly by

$$\varphi_M = \mathrm{A}^{-1}\varphi$$

for $\mathrm{A} \in \mathrm{GL}_m(\mathrm{B}_{\mathrm{R},[1,1]})$. Under this description of a vector bundle, a twist by $\mathcal{O}(1)$ amounts to sending $\mathrm{A} \mapsto \mathrm{A}\pi$ (recall π is the uniformizer for E; we use ϖ for perfectoid spaces). Once this setup is done, Kedlaya-Liu manually check global generation by producing explicit elements and verify $\varphi - \mathrm{A}$ is surjective after an appropriate twist to manipulate the matrix entries.

More precisely, they show that for $1 < r \leq q$ rational there are m elements

$$v_1, \ldots, v_m \in (\mathrm{B}^m_{\mathrm{R},[1,q]})^{\varphi=\mathrm{A}} = \mathrm{H}^0(\mathrm{X}_S, \mathcal{E})$$

which form a basis of $\mathrm{B}^m_{\mathrm{R},[r,q]}$. Applying this to enough strips proves global generation, and one proves this by showing $\varphi - \mathrm{A}$ is surjective in an *effective* way, that is one can pick preimages for $\varphi - \mathrm{A} : \mathrm{B}^m_{\mathrm{R},[1,q]} \to \mathrm{B}^m_{\mathrm{R},[1,1]}$ such that the preimage has a small spectral norm on $\mathrm{B}^m_{\mathrm{R},[r,q]}$. Kedlaya-Liu provide a convergent procedure to produce these preimages, and then pick $v_i = [\varpi]^M e_i - v'_i$ as small perturbation of the standard basis to land in the $\varphi = \mathrm{A}$ invariants. Here, v'_i is chosen so $(\varphi - \mathrm{A})(v'_i) = (\varphi - \mathrm{A})([\varpi]^M e_i)$ (thus landing in the $\varphi = \mathrm{A}$ fixed points) but has a sufficiently small norm on $\mathrm{B}^m_{\mathrm{R},[1,q]}$ so that these remain a basis. □

3. The HN formalism

We will begin by recalling what the Harder-Narasimhan formalism is for a curve $\mathrm{X}/\mathbf{C}$.

> Definition 3.1. Let $\mathcal{E}$ be a vector bundle on a smooth projective curve $\mathrm{X}/\mathbf{C}$. We define the *slope* of $\mathcal{E}$ to be $\lambda = \deg(\mathcal{E})/\mathrm{rank}(\mathcal{E}) \in \mathbf{Q}$.
>
> A vector bundle is *semistable* if any proper nonzero subbundle $\mathcal{E}'$ has $\lambda(\mathcal{E}') \leq \lambda(\mathcal{E})$.

> Theorem 3.2. Let $\mathcal{E}$ be a vector bundle on a smooth projective curve $\mathrm{X}/\mathbf{C}$. Then there exists a unique filtration
>
> $$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$
>
> such that all subquotients $\mathcal{F}_i = \mathcal{E}_{i+1}/\mathcal{E}_i$ are semistable and slopes of $\mathcal{F}_i$ decrease as the index i increases.

As it turns out, an extremely similar formalism can be defined on the Fargues-Fontaine curve X_C. The non-obvious part of the definition of a slope is defining the degree, which requires us to determine the line bundles.

PROPOSITION 3.3. Let $S^\#$ be a characteristic zero untilt lying over E_∞, the completion of the maximal abelian extension of E. Then there is an exact sequence of $\mathcal{O}_{X_S}$-modules

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{O}_{S^\#} \longrightarrow 0.$$

Sketch. This is used several times, so I will explain how to write down the maps.

Providing a map $\mathcal{O} \to \mathcal{O}(1)$ amounts to taking the data of an untilt $S^\#$ and then providing a section $s \in H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1))$. Using a slight modification of the Čech covering we used to show $\mathcal{O}(1)$ is ample, we can identify

$$H^0(X_{S,E}, \mathcal{O}_{X_{C,E}}(1)) = \mathcal{O}(Y_{[1,\infty]})^{\varphi=\pi}$$

where $Y_{[1,\infty]} = \{|[\varpi]| \leq |\pi| \neq 0\} \subset \operatorname{Spa} W_{\mathcal{O}_E}(S^+)$. Note that this is not contained in Y. Using the fact that Frobenius scales the radius function by q, we can further identify

$$\mathcal{O}(Y_{[1,\infty]})^{\varphi=\pi} = (B^+_{\text{cris}})^{\varphi=\pi}.$$

Now apply Scholze-Weinstein Theorem A: the Dieudonné functor on semiperfect rings is fully faithful. We obtain

$$H^0(X_{S,E}, \mathcal{O}_{X_{s,E}}(1)) = \operatorname{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(S^{\#+}/\pi))[1/\pi] = \tilde{G}(S^{\#+}/\pi) = \tilde{G}(S^{\#+})$$

where $G \simeq \operatorname{Spf} \mathcal{O}_E[[X]]$ is the Lubin-Tate formal group of E and $\tilde{E} = \varprojlim_{\times\pi} G \simeq \operatorname{Spf} \mathcal{O}_E[[\tilde{X}^{1/p^\infty}]]$ is the universal cover. In particular, the first identification we use the p-divisible group $E/\mathcal{O}_E$ and identify $G(S^{\#+}/\pi)$ with the points of the associated p-divisible group (by taking the p-adic Tate module for the formal group).

With this machinery in place, so long as our untilt lies over E_∞ we can produce a distinguished element of $\tilde{G}(C^{\#,+})$ via the map

$$V_\pi(G) \to \tilde{G}$$

where V_π is the rational π-adic Tate module. This arises by taking universal covers on $\bigcup_n G[\pi^n] \to G$. Given an untilt $C^\#/E_\infty$, we can produce an element of V_π which we use for the section.

The final map is just evaluation at $C^\#$. Exactness ends up being possible to reduce to $C^\#$ to the universal case of E_∞ where it can be checked directly. □

PROPOSITION 3.4. Let x be a characteristic zero untilt. The scheme $X_C^{\text{alg}} - [x]$ is affine, and the spectrum of a PID.

Now we can prove the following.

PROPOSITION 3.5. We have

$$\mathbf{Z} \simeq \mathrm{Pic}(\mathrm{X}_C)$$

via $n \mapsto \mathcal{O}(n)$.

Proof. First, by GAGA we may instead consider the algebraic curve. The corollary shows any vector bundle on $\mathrm{X}_C^{\mathrm{alg}}$ is trivialized on $\mathrm{X}_C^{\mathrm{alg}} - [x]$, so any vector bundle is of the form $\mathcal{O}(n[x])$. Here we are appealing to the fact that the local ring at x is a DVR, so by Beauville-Laszlo gluing we have

$$\mathrm{Pic}(\mathrm{X}_C^{\mathrm{alg}}) \simeq \mathrm{Pic}(\mathrm{X}_C^{\mathrm{alg}} - [x]) \times_{\mathrm{Pic}(\mathrm{D}_x^\circ)} \mathrm{Pic}(\mathrm{D}_x)$$

where $\mathrm{D}_x = \widehat{\mathcal{O}_{\mathrm{X}_C,x}}$ and D° punctures this. Knowing the local ring is a DVR, we get $\mathbf{Z}$. For example, if we had $\mathbf{C}_p[[t]]$ we look at $\mathbf{C}_p[[t]]$ lattices in $\mathbf{C}_p((t))$, which are classified by t^n. In general if R is a DVR with fraction field K these are going to be classified by $K^\times/\mathrm{R}^\times$, or the value group, which is $\mathbf{Z}$.

As this point, we already know $\mathrm{Pic}(\mathrm{X}_C) \simeq \mathbf{Z}$, but the isomorphism is not canonical.

It suffices to show $\mathcal{O}([x]) \simeq \mathcal{O}(1)$ for any untilt $x = \mathrm{Spa}(C^\sharp, C^{\sharp,+})$ to make the isomorphism canonical.

To see this look at the previous exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{X}_C} \longrightarrow \mathcal{O}_{\mathrm{X}_C}(1) \longrightarrow \mathcal{O}_{C\#} \longrightarrow 0.$$

This means that the map $\mathcal{O}_{\mathrm{X}_C} \to \mathcal{O}_{\mathrm{X}_C}(1)$ factors through the twisted ideal sheaf $\mathrm{I}_{[x]}(1)$, and by exactness it is an isomorphism as $\mathrm{I}_{[x]}(1) = \ker(\mathcal{O}_{\mathrm{X}_C}(1) \to \mathcal{O}_{C\#})$.

Observe $\mathrm{I}_{[x]}$ is just $\mathcal{O}(-[x])$. As we just argued that

$$\mathrm{I}_{[x]}(1) \simeq \mathcal{O}_{\mathrm{X}_C}$$

so in particular $\mathcal{O}(-[x]) \simeq \mathcal{O}_{\mathrm{X}_C}(-1)$ by untwisting. Taking duals, the claim follows. □

DEFINITION 3.6. Let $\mathcal{E}$ be a vector bundle on X_C. We define $\deg(\mathcal{E}) = \deg(\det \mathcal{E})$, where deg is the isomorphism $\mathrm{Pic}(\mathrm{X}_C) \to \mathbf{Z}$.

Then we set the slope λ to be the degree over the rank.

One can axiomatize a Harder-Narasimhan formalism and verify the axioms hold to deduce that it holds for X_C given this definition of a slope. This can be generalized, but the definition below is sufficient.

DEFINITION 3.7 (Abstract HN formalism). An abstract HN formalism consists of a quasi-abelian category $\mathcal{C}$ equipped with degree and rank functions $|\mathcal{C}| \to \mathbf{Z}_{\geq 0}$ which are additive on exact sequences.

REMARK 3.8. If $\mathrm{X} = \operatorname{Spec} \mathrm{R}$ is a scheme, generally $\mathsf{Vect}(\mathrm{X})$ is not quasi-abelian since we don't need to have kernels and cokernels (the third condition is that Ext is bifunctorial). But in the case that R is a Dedekind domain, like with a curve, this is true.

It's easy to check rank and degree are additive in short exact sequences. For rank this is clear, and for degree we just take determinant bundles: the determinant functor factors through $\mathrm{K}^0(\mathsf{Vect}(\mathrm{X}_C))$, so in particular for an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ we obtain $\det(\mathcal{E}_2) \simeq \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_3)$, and hence the degree is additive.

We then obtain the following corollary.

COROLLARY 3.9. The scheme $\mathrm{X}_C^{\mathrm{alg}}$ has a Harder-Narasimhan formalism. That is, Theorem 3.2 holds verbatim with the definition of semistable being the same.

REMARK 3.10. The vector bundle $\mathcal{O}(\lambda)$ is always stable of slope λ, and $\mathcal{O}(\lambda)^n$ is always semistable of slope λ, or lies in $\mathsf{Vect}^\lambda(\mathrm{X}_{C,\mathrm{E}})$.

REMARK 3.11. If $\mathcal{E} \simeq \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}(\lambda)^{n_\lambda}$, the slope of $\mathcal{E}$ is the n_λ-weighted average of the λ's that appear.

4. Reductions for the main theorem

Having now established the Harder-Narasimhan formalism on $\mathsf{Vect}(\mathrm{X}_C)$, we will be able to reduce the desired classification theorem to the case of semistable vector bundles and further to semistable slope 0 vector bundles. The triviality of semistable slope 0 vector bundles will be the most difficult part.

PROPOSITION 4.1. The cohomology group

$$\mathrm{H}^1(\mathrm{X}_S, \mathcal{O}_{\mathrm{X}_S}(\lambda))$$

is trivial for $\lambda \in \mathbf{Q}_{>0}$. In particular, $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\lambda')) = 0$ when $\lambda > \lambda'$.

Proof. This can be proven directly. First, we make a small reduction: if $\lambda = s/r$, replacing E with a degree E extension gives a covering $f : \mathrm{X}_{S,\mathrm{E}'} \to \mathrm{X}_{S,\mathrm{E}}$ of the curve where $f_*\mathcal{O}(s) = \mathcal{O}(s/r)$. Then it suffices to show vanishing for H^1 of $\mathcal{O}(n)$.

Recalling the module setup used to prove $\mathrm{H}^1(\mathrm{X}_C, \mathcal{E}(n)) = 0$ for $n \gg 0$, the corresponding object for $\mathcal{O}(n)$ is a free rank 1 module M over $\mathrm{B}_{C,[1,q]}$ equipped with

$$\varphi_M = \mathrm{A}^{-1}\varphi : M_{[q,q]} \to M_{[1,1]}.$$

Here A is an automorphism of $\mathrm{B}_{R,[1,1]}$. Recall that using this presentation of X_S we get H^0 as the $\varphi = \mathrm{A}$ invariants, since we need $M_{[q,q]}$ and $M_{[1,1]}$ to be identified. For higher cohomology we look at the derived invariants.

Since twisting corresponds to multiplication by π, we're looking at $\mathrm{A} = \pi^n$. To get H^1 to vanish we'll need to show

$$\varphi - \pi^n : \mathrm{B}_{C,[1,q]} \to \mathrm{B}_{C,[1,1]}$$

is a surjection.

This can be done fairly directly, without the more involved methods Kedlaya-Liu used for surjectivity. Any element of $\mathrm{B}_{C,[1,1]}$ has a decomposition into $\mathrm{B}_{C,[0,1]}[1/\pi]$ and $[\varpi]\mathrm{B}_{C,[1,\infty]}$. Here,

$$\mathrm{Y}_{C,[0,1]} = \{|\pi| \leq |[\varpi]| \neq 0\}$$

and $[1,\infty]$ does the reverse; $[1,1]$ asks for equality, which is why we have the decomposition.

Assume $f \in \mathrm{B}_{C,[0,1]}$. Then

$$g = \varphi^{-1}(f) + \pi^n\varphi^{-2}(f) + \pi^{2n}\varphi^{-3}(f) + \dots$$

converges in $\mathrm{B}_{[0,q]}$. Then g is an explicit preimage for f; similarly this works for $\mathrm{B}_{C,[0,1]}[1/\pi]$. For $[\varpi]\mathrm{B}_{C,[1,\infty]}$ we use

$$g = -\pi^{-n}f - \pi^{-2n}\varphi(f) - \dots$$

which converges in $\mathrm{B}_{C,[1,q]}$. Thus, we get explicit preimages.

To see the second claim, suppose we have an extension

$$0 \longrightarrow \mathcal{O}(\lambda) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(\lambda') \longrightarrow 0.$$

Then $\mathrm{H}^1(\mathrm{X}_C, \mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda')^\vee)$ parameterizes extensions. To see this is 0, it suffices to see $\mathrm{H}^1(\mathrm{X}_C, \mathcal{O}(\lambda - \lambda')) = 0$. This is precisely what the first claim says. ☐

Proposition 4.2. We have

$$\mathrm{Ext}^1(\mathcal{O}_{\mathrm{X}_C}(\lambda), \mathcal{O}_{\mathrm{X}_C}(\lambda)) = 0$$

for any $\lambda \in \mathbf{Q}$.

Proof. As seen in the previous proposition, this amounts to $\mathrm{H}^1(\mathrm{X}_C, \mathcal{O})$ vanishing.

We can again appeal to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{X}_S} \longrightarrow \mathcal{O}_{\mathrm{X}_S}(1) \longrightarrow \mathcal{O}_{S^\#} \longrightarrow 0$$

for an untilt. In this case after taking cohomology we get

$$\mathrm{H}^0(\mathrm{X}_S, \mathcal{O}_{\mathrm{X}_S}(1)) \xrightarrow{\log} S^\# \longrightarrow \mathrm{H}^1(\mathrm{X}_S, \mathcal{O}_{\mathrm{X}_S})$$

where the first map is given by the logarithm map

$$\tilde{\mathrm{G}}(S^{\#+}) \to \mathrm{G}(S^{\#+}) \to S^\#$$

where G is the Lubin-Tate formal group and $\tilde{\mathrm{G}}$ is the universal cover. Once we have identified $\tilde{\mathrm{G}}(S^{\#+})$ with global sections of $\mathcal{O}(1)$ via Scholze-Weinstein theorem A, Lemma 3.5.1 shows compatibility with the quasilogarithm. Unwinding definitions shows explicitly what the map to $S^\#$ is, and this map is pro-étale locally surjective with kernel $\underline{\mathrm{E}}$. This shows $\mathrm{H}^1(\mathrm{X}_C, \mathcal{O})$ vanishes pro-étale locally, but it's already a sheaf so it just vanishes. □

COROLLARY 4.3. To deduce $\mathcal{E} \simeq \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}(\lambda)^{n_\lambda}$ for any $\mathcal{E} \in \mathsf{Vect}(\mathrm{X}_{C,\mathrm{E}})$, it suffices to prove any semistable slope 0 vector bundle admits an injective map $\mathcal{O} \to \mathcal{E}$.

Proof. We argue by induction on the rank. Having computed $\mathrm{Pic}(\mathrm{X}_C) \simeq \mathbf{Z}$ via $n \mapsto \mathcal{O}_{X_C}(n)$, we know the rank one case is done.

Next, suppose the theorem is proven for rank n and let $\mathcal{E}$ be of rank $n+1$. If $\mathcal{E}$ is *not* semistable, then looking at the HN filtration

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

we know $r - 1 \neq 0$. Thus, we look at $\mathcal{E}_{r-1}$, knowing that $\mathcal{E}/\mathcal{E}_{r-1}$ is a semistable vector bundle. That is, we obtain an extension

$$0 \longrightarrow \mathcal{E}_{r-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{E}_{r-1} \to 0,$$

where by induction on both sides the vector bundles are a direct sum of $\mathcal{O}(\lambda)'s$ (and on the right, only the minimal slope λ). The first proposition then suffices to show $\mathcal{E}$ is a direct sum of $\mathcal{O}(\lambda)$'s.

Thus, we are reduced to the case where $\mathcal{E}$ is semistable of slope λ. By the second proposition, to deduce the claim amounts to producing an injective map

$$\mathcal{O}(\lambda) \to \mathcal{E}$$

since the category of semistable slope λ vector bundles is abelian (this is true in a general Harder-Narasimhan formalism); by applying the induction hypothesis, we see $\mathcal{E}$ is an extension of $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda)^{\mathrm{rank}(\mathcal{E})-1}$, which is necessarily trivial.

Finally, we reduce to the semistable slope 0 case. Let $\lambda = \frac{s}{r}$, and put E' as the unramified degree r extension of E. Consider the degree r covering

$$f : \mathrm{X}_{C,\mathrm{E}'} \to \mathrm{X}_{C,\mathrm{E}}.$$

Then $\mathcal{O}(\lambda) = f_*\mathcal{O}(s)$, so by adjunction we need a nonzero map

$$\mathcal{O}_{\mathrm{X}_{C,\mathrm{E}'}}(s) \to f^*\mathcal{E}.$$

Then by twisting we reduce to the slope 0 case. □

Thus we are left with proving the following theorem, which is where the technical details hide.

> THEOREM 4.4. Let $\mathcal{E} \in \mathsf{Vect}(\mathrm{X}_{C,\mathrm{E}})$ be semistable of slope 0. Then there exists an injective map
>
> $$\mathcal{O}_{\mathrm{X}_{C,\mathrm{E}}} \to \mathcal{E}.$$

5. DIAMONDS AND THE v-TOPOLOGY

To prove this final reduction, we will need some preliminary definitions. I will assume familiarity with adic and perfectoid spaces.

The first result is a useful motivational theorem.

> THEOREM 5.1 (Scholze). Let $\mathrm{X}/\mathbf{Q}_p$ be a rigid analytic variety. Then perfectoid spaces over X form a basis for the proétale topology.

For example, if X is "small" in the sense that there is an étale map $\mathrm{X} \to \mathrm{T}^n$, we can use the perfectoid torus $\tilde{\mathrm{T}}^n$ to give a proétale cover.

In fact, this is even more strongly the case: picking a proétale cover $\tilde{\mathrm{X}}$ which is perfectoid, we have

$$\mathrm{X} = \mathrm{Coeq}(\tilde{\mathrm{X}} \times_{\mathrm{X}} \tilde{\mathrm{X}} \rightrightarrows \tilde{\mathrm{X}})$$

in the category of analytic adic spaces.

Now observing this is a coequalizer of perfectoid spaces, the idea is that the diamond $X^\diamond$ should generalize the tilting construction to rigid analytic spaces over $\mathbf{Q}_p$. Indeed, once we have this presentation the assignment

$$X \mapsto X^\diamond$$

should forget the structure map to Spa $\mathbf{Q}_p$ by taking such a coequalizer presentation and tilting the perfectoid spaces.

This is made precise with the following definitions.

DEFINITION 5.2. Let Perf be the category of all characteristic p perfectoid spaces. A diamond D is a proétale sheaf on Perf such that

$$D = X/R$$

where $X \in$ Perf and $R \subset X \times X$ is an equivalence relation such that the projections onto each copy of X are proétale.

THEOREM 5.3 (Scholze). The category of diamonds has all products, fiber products, and quotients by pro-étale equivalence relations.

REMARK 5.4. We're using that the absolute product of characteristic p perfectoid spaces is again perfectoid.

To $X/\operatorname{Spa}\mathbf{Q}_p$ a rigid analytic space, using the previous coequalizer presentation we'd like to write

$$X^\diamond = \operatorname{Coeq}((\tilde{X} \times_X \tilde{X})^\flat \rightrightarrows \tilde{X}^\flat).$$

This doesn't literally make sense in adic space, but in the category of diamonds is does by definition. Indeed, if one interprets $(-)^\flat$ on a perfectoid space to mean the proétale $h_{(-)^\flat}$ given by the Yoneda embedding, this can be interpreted as a coequalizer in the category of diamonds. This now exists by construction.

However, it's unclear that this construction is independent of choices. A better construction is the following.

DEFINITION 5.5. Let $\mathrm{X}/\operatorname{Spa}\mathbf{Q}_p$ be a rigid analytic space. The presheaf $\mathrm{X}^\diamond$ on Perf is given by
$$S \mapsto \{(S^\#, S^\# \to \mathrm{X})\}.$$
Here, $S^\#$ is a characteristic zero untilt.

REMARK 5.6. If X is perfectoid and we try this, we get the sheaf for $\mathrm{X}^\flat$ under the Yoneda embedding for Perf. This is because $\mathsf{Perfd}_{\mathrm{X}} \simeq \mathsf{Perfd}_{\mathrm{X}^\flat}$.

REMARK 5.7. We have $\mathrm{Y}_{S,\mathrm{E}} = S \times (\operatorname{Spa}\mathrm{E})^\diamond$, and $\mathrm{X}_{S,\mathrm{E}} = S/\varphi^{\mathbf{Z}} \times (\operatorname{Spa}\mathrm{E})^\diamond$.

THEOREM 5.8 (Scholze). The presheaf $\mathrm{X}^\diamond$ is a proétale sheaf on Perf, and furthermore is a diamond. The following hold when X/K is a smooth rigid analytic space over a p-adic field:

- The functor
$$\mathrm{X} \mapsto (\mathrm{X}^\diamond, \mathrm{X}^\diamond \to \operatorname{Spa} K^\diamond)$$
is fully faithful.
- We can recover $|\mathrm{X}|$ through a presentation of the diamond via a perfectoid proétale cover $\tilde{\mathrm{X}}$ to obtain a proétale equivalence relation $\mathrm{R} = \tilde{\mathrm{X}} \times_{\mathrm{X}} \tilde{\mathrm{X}}$. One has $|\mathrm{X}| = |\tilde{\mathrm{X}}|/|\mathrm{R}|$.
- The category $\mathrm{X}^\diamond_{\text{ét}}$ of diamonds étale over $\mathrm{X}^\diamond$ recovers the usual site $\mathrm{X}_{\text{ét}}$.[a]

[a]One defines the diamond site $\mathrm{X}^\diamond_{\text{ét}}$ by saying $f : \mathcal{G} \to \mathcal{F}$ is étale if for $\mathrm{Y} \to \mathcal{F}$ perfectoid the pullback $\mathcal{G} \times_{\mathcal{F}} \mathrm{Y}$ is representable by a perfectoid space étale over Y.

We will also need to make use of a related concept called the v-topology on Perfd (we now use all perfectoid spaces).

DEFINITION 5.9. The v-topology on Perfd is the Grothendieck topology generated by open covers and all surjective maps of affinoids.

Initially it seems nothing could possibly be a sheaf for the v-topology, but actually many useful things are, including all diamonds.

THEOREM 5.10 (Scholze). Any diamond is a v-sheaf (regarded on Perf).

REMARK 5.11. Noting the similarity of the definition of a diamond and an algebraic space, this mirrors the result that algebraic spaces are automatically fpqc sheaves.

6. PROOF OF THE MAIN THEOREM

Recall that we proved that $\mathcal{E} \in \mathsf{Vect}(\mathrm{X}_C)$ admitting a decomposition $\mathcal{E} \simeq \bigoplus_{\lambda\in\mathbf{Q}} \mathcal{O}(\lambda)^{n_\lambda}$ is implies by the following theorem, which we will now go ahead and prove.

THEOREM 6.1. Let $\mathcal{E} \in \mathsf{Vect}(\mathrm{X}_{C,\mathrm{E}})$ be semistable of slope 0. Then there exists an injective map

$$\mathcal{O}_{\mathrm{X}_{C,\mathrm{E}}} \to \mathcal{E}.$$

Proof. We will break this proof up into several steps:

- Show that we can replace C by an extension.
- Show that after extending C, there exists $d \geq 0$ such that

$$\mathcal{O}_{\mathrm{X}_S}(-d) \to \mathcal{E}$$

  is injective.
- Reduce ruling out $d \geq 2$ to the key lemma.
- Reduce ruling out $d = 1$ to the key lemma.
- Prove the key lemma.

Step 1. Suppose the claim is true over C'/C. Considering the v-sheaf

$$S \in \mathsf{Perfd}_C \mapsto \{\mathcal{E}_S \simeq \mathcal{O}_{\mathrm{X}_S}^n\},$$

observe that since $\Gamma_{\text{proét}}(S, \underline{\mathrm{E}}) \simeq \Gamma(\mathrm{X}_S, \mathcal{O}_{\mathrm{X}_S})$ this is a v-quasitorsor for $\underline{\mathrm{GL}_n(\mathrm{E})}$. Indeed on S, any continuous map $|S| \to \mathrm{GL}_n(\mathrm{E})$ yields an automorphism of $\underline{\mathrm{E}}^n(S) = \mathrm{Hom}_{\text{cont}}(|S|, \mathrm{E}^n)$. Hence we obtain an action of $\underline{\mathrm{GL}_n(\mathrm{E})}(S)$ on $\Gamma_{\text{proét}}(S, \underline{\mathrm{E}}^n) \simeq \Gamma(\mathrm{X}_S, \mathcal{O}_{\mathrm{X}_S}^n)$, which gives the desired action. This is only a quasitorsor because we lack v-local triviality.

If the claim is true over C', then over C' there's a nonzero section (trivializing $\mathcal{E}$). This implies that over C we get an actual v-torsor, as we can deduce the v-local trivialization condition by the fact that $\operatorname{Spa} C' \to \operatorname{Spa} C$ is a v-cover.

Then in Scholze-Weinstein's Berkeley lectures it was shown any such $\underline{\mathrm{GL}_n(\mathrm{E})}$-torsor is representable by a perfectoid space pro-étale over $\operatorname{Spa} C$ (since $\mathrm{GL}_n(\mathrm{E})$ is locally profinite). This implies the torsor admits a section over C, so the claim follows.

Step 2. This is where we use that $\mathcal{O}_{X_C}(1)$ is ample! Let $\mathcal{L}$ be the sub line bundle of maximal degree. Since $\mathcal{E}$ is semistable of slope zero, the degree of $\mathcal{L}$ is ≤ 0. Thus, if $\mathcal{L}$ simply exists, we obtain an injection $\mathcal{O}_{X_S}(-d) \to \mathcal{E}$. In particular, all we need is for $\mathcal{E}$ to admit a global section after a twist. This indeed is the case by ampleness. Once we know $\mathcal{E}$ admits a sub line bundle, we can just take the maximal degree one.

Introduction of the key lemma. If $d = 0$, we are done. We will reduce cases where $d > 0$ to the key lemma below, which gives a global section and hence the desired map $\mathcal{O}_{X_{\mathbf{C},E}} \to \mathcal{E}$. We use step (1) to be able to take the extension to satisfy this hypothesis.

> LEMMA 6.2 (Key lemma). Let
>
> $$0 \longrightarrow \mathcal{O}_{X_C}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X_C}(1/n) \longrightarrow 0$$
>
> be an extension of vector bundles with $n \geq 1$. Then after taking some extension of C, $\mathcal{E}$ admits a global section.

Step 3. The idea is that having $d \geq 2$ contradicts minimality of d if we assume the key lemma. Since we chose the minimal d, $\mathcal{F} = \mathcal{E}/\mathcal{O}(-d)$ is again a vector bundle. It has rank $\leq n-1$, degree d and positive slope.

Thus, using the main theorem inductively, we'll get an injection $\mathcal{O}(-d+2) \to \mathcal{F}$ since $\mathcal{O}(-d+2)$ has maps to $\mathcal{O}(\lambda)$ for any $\lambda \geq 0$ (recall we made a map $\mathcal{O} \to \mathcal{O}(1)$, hence to $\mathcal{O}(n)$, and $\mathcal{O}(\lambda)$ by changing E). If $d = 1$, it's possible $\mathcal{F} = \mathcal{O}(1/(n-1))$ and we won't get a map from $\mathcal{O}(1)$ (unless $n = 2$; in many notes one skips straight to ruling out $d \geq 1$ by assuming this).

Now we apply this injection. Pulling back

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

by the morphism induces an extension

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(-d-2) \longrightarrow 0.$$

By the key lemma, after twisting to get an extension of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ after enlarging C we obtain an injection $\mathcal{O} \to \mathcal{G}(d-1)$, and hence an injection

$$\mathcal{O}(-d+1) \to \mathcal{G} \to \mathcal{E}$$

contradicting minimality.

Step 4. Suppose that in step 2 we obtained $d = 1$. We then get an extension

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ has rank $\leq n-1$, degree 1, and slope ≥ 0. Via induction we can apply the classification theorem, telling us

$$\mathcal{F} \simeq \mathcal{O}^i \oplus \mathcal{O}\left(\frac{1}{n-1-i}\right).$$

If $i = 0$, by the key lemma we are done. If $i \neq 0$, then pick a map $\mathcal{O} \to \mathcal{F}$ and pull back by this. Then we can apply the classification theorem on the pullback $\mathcal{E}'$ of $\mathcal{E}$ by the map, deducing that we have an injection $\mathcal{O} \to \mathcal{E}' \to \mathcal{E}$.

Step 5. It remains to prove the key lemma. We're given an extension

$$0 \longrightarrow \mathcal{O}_{\mathrm{X}_C}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathrm{X}_C}(1/n) \longrightarrow 0$$

and wish to show after taking an extension of C that $\mathcal{E}$ admits a global section. To avoid introducing details about diamonds, I will only give a brief sketch of the idea here. Taking cohomology of this exact sequence, we obtain an injection

$$\mathrm{BC}(\mathcal{O}(1/n)) \to \mathrm{BC}(\mathcal{O}(-1)[1])$$

of Banach-Colmez spaces.

One can show that $\mathrm{BC}(\mathcal{O}(1/n))$ is a perfectoid disk $\tilde{\mathrm{D}}_C$ and

$$\mathrm{BC}(\mathcal{O}(-1)[1]) \simeq (\mathbf{A}^1_{C\#})^\diamond/\underline{\mathrm{E}}.$$

However, we can argue that after base extension to C'/C

$$\tilde{\mathrm{D}}_C \to (\mathbf{A}^1_{C\#})^\diamond/\underline{\mathrm{E}}$$

is necessarily surjective, implying the map is an isomorphism. Indeed, the image hits a non-classical point (in the target classical points are totally disconnected, but the source is connected and not a point); this means after base extension the image contains a non-empty open subset of the diamond $\mathrm{BC}(\mathcal{O}(-1)[1])$.

That is, the image of the map contains an open neighborhood of the origin of the afine line in $(\mathbf{A}^1_{C\#})^\diamond$ after base extension, which due to the scaling action of $\mathrm{E}^\times$ implies surjectivity.

But this cannot be the case, as it would imply the map is an isomorphism. The target $\mathrm{BC}(\mathcal{O}(-1)[1])$ is not representable but the source is by a perfectoid disk, and representability is by definition preserved under isomorphisms of diamonds.

> Remark 6.3. Given that this decomposition holds for isomorphism classes, it's a natural question to ask what exactly the difference in the categories is conceptually.

The easy answer is that some morphisms are different: in isocrystals, $\mathrm{Hom}(\mathrm{V}_0, \mathrm{V}_{-1}) = 0$ but $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ is nonempty (as we saw with the exact sequence!).

However, there is a more interesting answer that drops any reference to isocrystals: it turns out with a modification of the curve to an "absolute curve", we literally get an equivalence. We can contemplate the category

$$\mathrm{Bun}_{\mathrm{FF}}(\mathrm{X})$$

for any v-stack X of morphisms of v-stacks $\mathrm{X} \to \mathrm{Bun}_{\mathrm{FF}}$.

By a recent theorem of Anschütz, for $\mathrm{Spa}\, k^\diamond$ ($k = \overline{\mathbf{F}}_q$) we actually obtain

$$\mathrm{Isoc}_{\breve{\mathrm{E}}} \simeq \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spa}\, k^\diamond).$$

One should think of this as "vector bundles on the absolute curve $\mathrm{X}_{k,\mathrm{E}}$", even though such an object doesn't literally exist. The difference between the two categories then has to do with the difference between $\mathsf{Vect}(\mathrm{X}_{k,\mathrm{E}}) := \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spa}\, k^\diamond)$ and $\mathsf{Vect}(\mathrm{X}_{C,\mathrm{E}})$.

□