THE ARC TOPOLOGY

1. Definition and motivation

The arc topology is a Grothendieck topology on $\mathsf{Sch}_{\text{qcqs}}$ which is very fine but yet many useful constructions satisfy arc-descent.

In order, from coarse to fine, we have:

$$\text{Zariski} \to \text{étale} \to \text{fppf} \to \text{arc}.$$

The fact that the topology is so fine makes checking a statement arc-locally very powerful, since you can often reduce to a very simple case.

Definition 1.1. An arc is a rank ≤ 1 valuation ring V. Letting $K = \text{Frac}(\mathrm{V})$, this means that we have a map

$$K^\times/\mathrm{V}^\times \hookrightarrow (\mathrm{R}, +)$$

where we order $K^\times/\mathrm{V}^\times$ in the usual way by saying $(\mathrm{V} \setminus 0)/\mathrm{V}^\times$ are the non-negative elements. In other words, the associated valuation can be regarded as landing inside of $(\mathrm{R}, +)$.

We will also refer to Spec V as an arc.

The reason for calling this an arc is that the typical example is something like $\text{Spec}\, \mathbf{C}[[x]]$, and so it is locally given by a single parameter.

With this definition in place, we can directly define what an arc-cover is.

Definition 1.2. A morphism $f : \mathrm{Y} \to \mathrm{X}$ of qcqs schemes is an arc-covering if for any morphism $\text{Spec}\,\mathrm{V} \to \mathrm{X}$ from any arc Spec V there is an extension $\text{Spec}\,\mathrm{W} \to \text{Spec}\,\mathrm{V}$ of arcs (an injective local homomorphism on the valuation rings, or faithfully flat morphism) such that we have a commutative diagram

$$\begin{tikzcd} \text{Spec}\,\mathrm{W} \arrow[r] \arrow[d] & \mathrm{Y} \arrow[d, "f"] \\ \text{Spec}\,\mathrm{V} \arrow[r] & \mathrm{X}. \end{tikzcd}$$

Visually, you should think of a morphism $Y \to X$ and asking that each arc in X has a lift to one in Y. With this comes the usual way we define a Grothendieck topology after defining the notion of a covering.

Definition 1.3. The arc-topology on $\mathsf{Sch}_{\mathrm{qcqs}}$ is the Grothendieck site with coverings

$$\{Y_i \xrightarrow{f_i} X\}_{i \in I}$$

such that for $U \subset X$ an affine open there exists a finite set J along with $\iota : J \to I$ and $U_j \subseteq f_{\iota(j)}^{-1}(U)$ for each $j \in J$ such that $\coprod_j U_j \to U$ is an arc-cover.

The restriction to qcqs schemes is somewhat superficial: a Zariski sheaf on Sch is uniquely determined $\mathsf{Sch}_{\mathrm{qcqs}}$, and Zariski descent is a mild condition satisfied by all relevant examples. By restricting to qcqs schemes, we are able to give more reasonable criteria for being a sheaf as the arc topology is finitary (meaning given a covering, a finite subset is a cover).

Definition 1.4. Let $\mathcal{C}$ be an ∞-category, in our case typically $D(\mathbf{Z})^{\geq 0}$. An arc-sheaf $\mathscr{F}$ is a presheaf $\mathsf{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \to \mathcal{C}$ such that

$$\mathscr{F}(X) \longrightarrow \varprojlim(\mathscr{F}(Y) \rightrightarrows \mathscr{F}(Y \times_X Y) \Rrightarrow \dots)$$

is an equivalence for all arc covers $Y \to X$ and further carries finite disjoint unions to finite products.

The reason we can make this as a definition is due to SAG A.3.3.1, which tells us this criterion for the sheaf condition for a finitary Grothendieck topology. Otherwise, a priori we need to check descent for every possible covering of $Y_i \to X$ according to the previous definition; this basically allow us to check just the condition on affine opens but globally. This is where a scheme being qcqs and finite disjoint unions come into play.

From the original definition, it can be immediately seen that the topology is very fine.

Lemma 1.5. Any proper and surjective map is an arc-cover. Any faithfully flat cover is an arc-cover.

Proof. For the first assertion, lift the generic point η of V first by surjectivity. Then the valuative criterion of properness allows us to lift the closed point of V as well, showing that we have an arc-cover.

For a faithfully flat cover, we first lift the closed point of V using surjectivity. Then we can lift the generic point η by using that flat morphisms can lift generizations (flat ring maps satisfy going down). $\square$

These two classes of arc-covers show that it is finer than most topologies you are used to.

However, it's important to note that being an arc-cover is more restrictive than simply being surjective.

> Example 1.6. Take the blowup $\tilde{\mathbf{A}}^2$ of $\mathbf{A}^2$ at the origin, and remove a point p in the exceptional locus of $\tilde{\mathbf{A}}^2$. Call the resulting scheme X. Then $X \to \mathbf{A}^2$ is still surjective, but it is not an arc-cover because an arc going in the direction corresponding to p does not have a lift.

The main idea for why we might consider using the arc-topology comes from the v-topology, which is closed related. Here v stands for *valuation*, and the definition is precisely the same on qcqs schemes as the arc-topology except the condition on ranks of valuation rings is removed. Many important functors we use satisfy v-descent already, such as étale cohomology.

On Noetherian schemes with finite type morphisms, the v-topology has a very geometric interpretation: it is generated by étale covers and proper surjections (which is why it's not surprising étale cohomology satisfies descent for it). Note also that it agrees with the arc topology on such schemes.

One of the main results about the arc topology is that it is not too hard to deduce from v-descent.

> Theorem 1.7. Let $\mathscr{F} : \mathsf{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \to D(\mathbf{Z})^{\geq 0}$ be a finitary functor. This means that filtered limits with affine transition maps on schemes (e.g. an inverse system of affine schemes) to filtered colimits.
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> Then the following are equivalent:
>
> (1) $\mathscr{F}$ is an arc sheaf.
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> (2) (AIC-v-excision) $\mathscr{F}$ is a v-sheaf, and for every valuation ring V with $\mathrm{Frac}(V)$ algebraically closed (called AIC) and $\mathfrak{p} \in \operatorname{Spec} V$ we have a cartesian square
>
> $$\begin{tikzcd} \mathscr{F}(\operatorname{Spec} V) \arrow[r] \arrow[d] & \mathscr{F}(\operatorname{Spec} V/\mathfrak{p}) \arrow[d] \\ \mathscr{F}(\operatorname{Spec} V_{\mathfrak{p}}) \arrow[r] & \mathscr{F}(\operatorname{Spec} \kappa(\mathfrak{p})). \end{tikzcd}$$

(3) (Milnor excision) $\mathscr{F}$ is a v-sheaf and sends any cartesian square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A/I \arrow[d] \\ B \arrow[r] & B/J \end{tikzcd}$$

(where necessarily $f : A \to B$ carries I isomorphically into J) to a cartesian square. Such squares are called Milnor squares.

Remark 1.8. We call this property excision because in topological spaces this refers to $H^*(X \setminus U, A \setminus U) \simeq H^*(X, A)$. This is equivalent to saying the square

$$\begin{tikzcd} H^*(X) \arrow[r] \arrow[d] & H^*(X \setminus U) \arrow[d] \\ H^*(A) \arrow[r] & H^*(A \setminus U) \end{tikzcd}$$

is cartesian, since the fibers of the downward maps are relative cohomologies and isomorphic. We are asking a similar thing here: the morphism f, scheme-theoretically, is an isomorphism on the open complement of B/J.

We will focus on showing (1) iff (2) first, as this is most important.

(1) implies (2). Let V be an AIC valuation ring. Then it is not difficult to verify

$$V \to V_{\mathfrak{p}} \times V/\mathfrak{p}$$

is an arc cover, where $\mathfrak{p} \in \operatorname{Spec} V$. We will use this to rewrite $\mathscr{F}(\operatorname{Spec} V)$.

One proves this morphism is an arc cover by showing *any* map $f : V \to W$ to a rank ≤ 1 valuation ring factors through $V_{\mathfrak{p}} \times V/\mathfrak{p}$. If it does, it must factor through one of the components by connectedness. If $f(\mathfrak{p}) = 0$ then it factors through $V/\mathfrak{p}$ and we are done. Otherwise, there's $x \in \mathfrak{p}$ so $f(x) \neq 0$, and we wish to show f factors over $V \to V_{\mathfrak{p}}$. We obtain a map induced by f as follows:

$$V_{\mathfrak{p}} \to V_{\mathfrak{p}}[1/x] = V[1/x] \to W[1/f(x)] \subset K = \operatorname{Frac}(W).$$

Let $W' \subset K$ be the W-subalgebra generated by the image. We have $f(x) \cdot W' \subseteq W$, since $x \cdot V_{\mathfrak{p}} \subseteq V$: given $\frac{a}{s}$ with $a \in V$ and $s \in V \setminus \mathfrak{p}$, multiplication by x makes this of the form $a \in \mathfrak{p}$ and $s \in V \setminus \mathfrak{p}$. But then $s|a$ (ideals totally ordered by inclusion in a valuation ring) and $a = sb$ for $b \in \mathfrak{p}$, we land in V.

Due to being rank ≤ 1 (Krull dimension = number of subrings of fraction field containing valuation ring), it follows $W' = W$. We then see $V_{\mathfrak{p}} \to V[1/x] \to W[1/f(x)]$ actually lands inside W, so the map $V \to W$ must factor through $V_{\mathfrak{p}}$. The claim follows.

Denote the target $V_{\mathfrak{p}} \times V/\mathfrak{p}$ by $\tilde{V}$. Then we claim the cosimplicial object

$$\mathscr{F}(\tilde{V}^{\otimes \bullet+1})$$

computes (after totalization) the fibre product $\mathscr{F}(V_{\mathfrak{p}}) \times_{\mathscr{F}(\kappa(\mathfrak{p}))} \mathscr{F}(V/\mathfrak{p})$. Here, we are tensoring $\tilde{V}$ over V.

In this setting $V_{\mathfrak{p}} \otimes_V V_{\mathfrak{p}} = V_{\mathfrak{p}}$, $V/\mathfrak{p} \otimes_V V/\mathfrak{p} = V/\mathfrak{p}$. More importantly, $\kappa(\mathfrak{p})$ will ultimately arise since $V_{\mathfrak{p}} \otimes_V V/\mathfrak{p} = \kappa(\mathfrak{p})$. Now we appeal to the fact that $\mathscr{F}$ preserves finite products to obtain for $\mathscr{F}(\tilde{V}^{\otimes \bullet+1})$

$$\mathscr{F}(V_p) \times \mathscr{F}(V/\mathfrak{p}) \rightrightarrows \mathscr{F}(V_{\mathfrak{p}}) \times \mathscr{F}(\kappa(\mathfrak{p}))^{\times 2} \times \mathscr{F}(V/\mathfrak{p}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

In general, we'll get a number of copies of $\mathscr{F}(\kappa(\mathfrak{p}))$ in between. If we were working in a 1-category, the claim is now clear since the fiber product $A \times_B C$ is just the equalizer

$$A \times B \rightrightarrows C$$

This computes the same thing as the diagram with $A \times C^{\times 2} \times B$ where the maps to A, B are the canonical projections and the maps to the copies of C are the same (projections plus the canonical map to C). Note that the two maps to $A \times C^{\times 2} \times B$ just swap the order, so the equalizer won't change.

In an ∞-category, it is more complicated. It is similar to but not quite identical to the totalization $A \times C^{\bullet} \times B$ computing $A \times_C B$, but the number of copies of C is different.

Once this is established, the claim is clear. We know that by arc-descent $\mathscr{F}(\tilde{V}^{\otimes \bullet+1})$ computes $\mathscr{F}(V)$, so then $\mathscr{F}(V_{\mathfrak{p}}) \times_{\mathscr{F}(\kappa(\mathfrak{p}))} \mathscr{F}(V/\mathfrak{p}) \simeq \mathscr{F}(V)$. But this is just telling us the AIC v-excision square we wanted is cartesian. $\square$

One thing to note is that v-sheaves can fail excision, so (2) implies (1) has content. Indeed, $V \to \tilde{V} = V_{\mathfrak{p}} \times V/\mathfrak{p}$ is *not* a v-cover in general, and in (1) implies (2) we showed descent for this map is equivalent to excision for AIC valuation rings. Take V to have rank ≥ 2 and let $\mathfrak{p}$ be nonzero and nonmaximal (possible due to the rank). Then $V \to \tilde{V}$ is not a v-cover. Assume we have an extension $V \to W$ of valuation rings. To rule out being a v-cover, it suffices to show that we cannot factor this as $V \to \tilde{V} \to W$ as this rules out lifting the identity map $\operatorname{Spec} V \to \operatorname{Spec} V$. Now $\operatorname{Spec} W \to \operatorname{Spec} V$ is surjective and $\operatorname{Spec} W$ is connected. Noting $\operatorname{Spec} \tilde{V}$ is a disjoint union of $\operatorname{Spec} V_{\mathfrak{p}}$ and $\operatorname{Spec} V/\mathfrak{p}$ and elements of $\mathfrak{m} \setminus \mathfrak{p}$ in $V_{\mathfrak{p}}$ are invertible (remember we need a local homomorphism of local rings) this has to factor as $V \to V/\mathfrak{p} \to W$. But there is a nontrivial kernel.

For (2) implies (1), we will need the following notion.

DEFINITION 1.9. Let $\mathscr{F}$ be a functor $\mathsf{Sch}^{\mathrm{op}}_{\mathrm{qcqs,R}} \to \mathrm{D}(\mathbf{Z})^{\geq 0}$. We say that a map $f : \mathrm{Y} \to \mathrm{X}$ of qcqs schemes over $\operatorname{Spec} \mathrm{R}$ is of $\mathscr{F}$-descent if the natural map

$$F(\mathrm{X}) \longrightarrow \varprojlim(F(\mathrm{Y}) \rightrightarrows F(\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \dots)$$

is an equivalence. It is of universal $\mathscr{F}$-descent if all base changes along morphisms to X of this morphism are of $\mathscr{F}$-descent.

DEFINITION 1.10. A family of morphisms $\{\mathrm{X}_i \to \mathrm{X}\}$ detects universal $\mathscr{F}$-descent when $f : \mathrm{Y} \to \mathrm{X}$ is of universal $\mathscr{F}$-descent if and only if all of the base changes $\mathrm{Y} \times_{\mathrm{X}} \mathrm{X}_i \to \mathrm{X}_i$ are of universal $\mathscr{F}$-descent.

Note that morphisms of universal $\mathscr{F}$-descent are stable under base change, so only one direction is interesting.

Sketch of (2) implies (1). Let $f : \mathrm{Y} \to \mathrm{X}$ be an arc cover. We need to check that $\mathscr{F}$ satisfies descent for f given that it is a v-sheaf and satisfies excision on AIC valuation rings.

The first reduction uses the following lemma:

LEMMA 1.11. Let A be a ring, and $\mathscr{F}$ a finitary v-sheaf. Then there is a family of AIC valuation rings V_i such that $\{f_i : \mathrm{A} \to \mathrm{V}_i\}$ detects universal $\mathscr{F}$-descent.

Since Zariski descent is given, we can now use this lemma and stability of arc covers under base change to reduce showing universal $\mathscr{F}$-descent for arc covers

$$\operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{V}$$

where V is an AIC valuation ring. We can write A as a filtered colimit of arc covers of V, and then use the fact that morphisms of universal $\mathscr{F}$-descent are preserved under filtered colimits of rings.

The key observation is then that ideals in $\operatorname{Spec} \mathrm{V}$ form a poset under inclusion, and each interval $\mathrm{I} = [\mathfrak{p}, \mathfrak{q}]$ has an associated AIC valuation ring $\operatorname{Spec}(\mathrm{V}/\mathfrak{p})_{\mathfrak{q}}$.

The idea is to work out way up from universal descent of $\mathrm{V}_{\mathrm{I}} \to \mathrm{A} \otimes_{\mathrm{V}} \mathrm{V}_{\mathrm{I}}$ when $\operatorname{len}(\mathrm{I}) \leq 1$ up to $\mathrm{V} \to \mathrm{A}$.

In this initial case where $\operatorname{len}(\mathrm{I}) \leq 1$, we use the crucial fact that a cover is an arc-cover *if and only if* all of its base changes to rank ≤ 1 valuation rings (e.g. V_{I} in this case) are v-covers. Noting v-covers are preserved under base change, we get universal descent when $\operatorname{len}(\mathrm{I}) \leq 1$.

We can make some simple verifications at this point, using that $V_I \to A \otimes_V V_I$ being a v-cover suffices for universal $\mathscr{F}$-descent.

- If $J \subset I$, since morphisms of universal $\mathscr{F}$-descent are stable under base change we get descent for V_J form V_I.
- If $\mathfrak{p}$ is not maximal, we can find $\mathfrak{q}$ so $I = [\mathfrak{p}, \mathfrak{q}]$ gives a v-cover $V_I \to A \otimes_V V_I$. Indeed, assuming there's no immediate successor to $\mathfrak{p}$ (already covered by $\mathrm{len}(I) \leq 1$) we have
$$\kappa(\mathfrak{p}) = \varinjlim_{I'=[\mathfrak{p},\mathfrak{q}']} V_{I'}$$
over $\mathfrak{q}' \supset \mathfrak{p}$. But $\kappa(\mathfrak{p}) \to A \otimes_V \kappa(\mathfrak{p})$ is a v-cover (by the original case) and from this one can deduce of the terms $V_{I'}$ gives a v-cover (this uses the finite presentation hypothesis on A).
- Similarly, we can do the same for finding $\mathfrak{p}$ when $\mathfrak{q} \neq 0$ so we have universal descent for $V_{[\mathfrak{p},\mathfrak{q}]} \to A \otimes_V V_{[\mathfrak{p},\mathfrak{q}]}$.

The key thing to putting these together in order to get $I = \operatorname{Spec} V$ is the ability to deduce universal descent for $V_{I \cup J}$ given that $I \cap J \neq \emptyset$ and universal descent for V_I, V_J. Combinatorially, this is enough to deduce the claim for $I = \operatorname{Spec} V$ and complete the argument.

By base change you can assume $I \cup J = \operatorname{Spec} V$, in which case we want to prove for any V-algebra B that

$$\begin{tikzcd}
\mathscr{F}(B) \arrow[r] \arrow[d] & \mathscr{F}(B \otimes_V V_I) \arrow[d] \\
\mathscr{F}(B \otimes_V V_J) \arrow[r] & \mathscr{F}(B \otimes_V V_{I \cap J}).
\end{tikzcd}$$

is cartesian. It then follows from formal properties of universal descent that the collection $\{V \to V_I, V \to V_J, V \to V_{I \cap J}\}$ detects universal $\mathscr{F}$-descent for $V \to B$. That is, we can deduce universal descent for $V \to B$ from universal descent for $V \to B \otimes V_I$, etc in the diagram, so we just need to put $B = A$. To see this, in $\mathscr{F}(B^{\otimes_V \bullet})$ since this cartesian square holds for all B we can write each of the terms as the appropriate fiber product. Then when computing the totalization, we use $\mathscr{F}$-descent for the terms in the fiber product and compatibility of fiber products and limits.

To prove this, one further reduces to the case where $I \cap J$ is a singleton using a 2/3 argument for cartesian squares. This is then possible to deduce from excision, which implies its variant
$$\mathscr{F}(B) \simeq \mathscr{F}(B \otimes_V V_{\mathfrak{p}}) \times_{\mathscr{F}(B \otimes_V \kappa(\mathfrak{p}))} \mathscr{F}(B \otimes_V V/\mathfrak{p}).$$

$\square$

(1) implies (3). We can reduce to the case of an AIC rank ≤ 1 valuation ring A after noting that we can check isomorphisms of arc sheaves on such rings and stability of Milnor squares under base change to an integral domain.

The exact same argument we used for the excision datum $(V \to V_{\mathfrak{p}}, \mathfrak{p})$ to argue any morphism $V \to W$ for W a rank ≤ 1 valuation ring factors through $V_{\mathfrak{p}} \times V/\mathfrak{p}$ shows for a general excision datum the same holds for maps $A \to W$ factoring as $A \to B \times A/I \to W$. Then in our excision datum $(A \to B, I)$ either $A \to A/I$ admits a section or $A \to B$ admits a section s via the identity map $A \to A$ as A is an AIC rank ≤ 1 valuation ring.

In the first case where $A \to A/I$ admits a section it's trivial as $I = 0$. In the second case, we do a 2/3 property argument with cartesian squares to reduce to the case $s : B \to A$ is surjective. That is, we get a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[d] \\ B \arrow[r] \arrow[d, two heads, "s"] & B/J \arrow[d, "s"] \\ A \arrow[r] & A/I \end{tikzcd}$$

where trivially the outer square is cartesian, so by the 2/3 property for cartesian squares the top square is cartesian if the bottom is. However, the bottom reduces us to checking Milnor squares with a surjective map of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads, "s"] & B/J \arrow[d, "s"] \\ A \arrow[r] & A/I \end{tikzcd}$$

are sent to cartesian squares.

In this scenario, Milnor excision now more easily holds. We can again localize to B being an AIC valuation ring by the same technique and then either $B \to A$ or $B \to B/J$ are isomorphisms (since ideals are totally ordered). $\square$

2. Proving arc-descent for etale cohomology

The main theorem I will be focusing on is proving arc-descent for étale cohomology, using the criterion in 1.7.

Theorem 2.1. Let Λ be a finite ring. Then the functor
$$X \mapsto R\Gamma(X_{\text{ét}}, \underline{\Lambda})$$
is a finitary arc-sheaf on $\mathsf{Sch}^{\text{op}}_{\text{qcqs}}$.

More generally, let $\mathscr{F}$ be a torsion étale sheaf on $(\operatorname{Spec} R)_{\text{ét}}$. Then the functor
$$(X, f : X \to \operatorname{Spec} R) \mapsto R\Gamma(X_{\text{ét}}, f^*\mathscr{F}) \in D(\Lambda)$$
is a finitary arc-sheaf on $\mathsf{Sch}^{\text{op}}_{\text{qcqs},R}$.

Following the theorem, the first step is going to be proving v-descent for étale cohomology.

Lemma 2.2. Fix a functor $F : \mathsf{Sch}^{\text{op}}_{\text{qcqs},R} \to D(\mathbf{Z})^{\geq 0}$, and let $f : Y \to X$ and $g : Z \to Y$ be morphisms in $\mathsf{Sch}^{\text{op}}_{\text{qcqs},R}$. Then if f has a section, it is of universal F-descent.

Proof. This means there is a morphism $s : Y \to X$ so $f \circ s = \text{id}$. This yields a splitting of the cosimplicial diagram
$$F(Y) \rightrightarrows F(Y \times_X Y) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$
which then implies descent. More concretely, reducing to the case of a ring you can write explicit contracting homotopies $B^{\otimes_A n} \to B^{\otimes_A (n-1)}$ via sending $b_1 \otimes \ldots \otimes b_n \mapsto s(b_1)b_2 \otimes \ldots b_n$. Upon base change, we just get a base changed section. $\square$

We are now ready to prove v-descent of étale cohomology.

Theorem 2.3. Let $\mathscr{F}$ be a torsion étale sheaf on $(\operatorname{Spec} R)_{\text{ét}}$. Then the functor
$$(X, f : X \to \operatorname{Spec} R) \mapsto R\Gamma(X_{\text{ét}}, f^*\mathscr{F})$$
satisfies v-descent for $\mathsf{Sch}^{\text{op}}_{\text{qcqs},R}$.

Proof. The idea comes from the earlier remark that the v-topology is not too different from the h-topology. Specifically, let $Y \to X$ be a v-cover of qcqs schemes. Since we are in the qcqs setting, we can write Y as a filtered limit of a tower of finitely presented X-schemes with affine transition maps.

Now étale cohomology will turn such a limit into a filtered colimit, we can reduce to the case that $Y \to X$ is finitely presented. If we had a Noetherian assumption, at this point we could reduce to verifying étale covers and proper surjections by comparing to the h-topology. The first class is obvious; the second is possible to tackle with proper base change for étale cohomology.

Fortunately without this assumption we can use a similar strategy. It is known that a finitely presented morphism of qcqs schemes can be factored into a quasi-compact open covering and a proper finitely presented surjection.

For a quasi-compact open covering, similar to étale covers, the descent is immediate. We are then left with tackling proper finitely presented surjections.

The étale topos has enough points, which means isomorphisms can be tested stalkwise. For descent, we want an isomorphism to the Cech-Alexander complex

$$\mathrm{R}\Gamma(\mathrm{X}_{\text{ét}}, f^*\mathscr{F}) \simeq \mathrm{R}\Gamma(\mathrm{Y}^{\bullet}_{\text{ét}}, f^*\mathscr{F})$$

in the case $\mathrm{Y} \to \mathrm{X}$ is a proper finitely presented surjection. The isomorphism can be tested stalkwise, so we may assume that X is the spectrum of a strictly Henselian local ring.

By proper base change, we have $\mathrm{R}\Gamma(\mathrm{Y}_{\text{ét}}, f^*\mathscr{F}) \simeq \mathrm{R}\Gamma((\mathrm{Y}_x)_{\text{ét}}, f^*\mathscr{F})$ where $x \in \mathrm{X}$ is the closed point with residue field $\kappa(x)$. By topological invariance of étale cohomology, we are free to base change to the algebraic closure without changing the value of the functor. However, after base change we obtain a section $\mathrm{Y}_x \times \overline{x} \to \overline{x}$, which implies universal descent. Thus, we get descent for the map $\mathrm{Y}_x \to x$, which proves the theorem. □

Note that the torsion hypothesis is largely present because proper base change requires it.

The next step is to verify excision.

Proposition 2.4. The étale cohomology functor

$$\mathrm{R}\Gamma_{\text{ét}} : \mathsf{Sch}^{\mathrm{op}}_{\mathrm{qcqs,R}} \to \mathrm{D}(\mathbf{Z})^{\geq 0}$$

sending $(\mathrm{X}, f : \mathrm{X} \to \operatorname{Spec} \mathrm{R}) \mapsto \mathrm{R}\Gamma(\mathrm{X}_{\text{ét}}, f^*\mathscr{F})$ satisfies AIC-v-excision.

Proof. To apply Theorem 1.7, we need to check étale cohomology is a finitary functor. This is true since étale cohomology commutes with filtered colimits of rings. It then suffices to check the AIC-v-excision condition.

We first note that an absolutely integrally closed valuation ring V, the type used for AIC-v-excision in Theorem 1.7, is strictly Henselian.

Let $\mathfrak{m} \subset \mathrm{V}$ be the maximal ideal. By virtue of being AIC, given a monic polynomial in $\mathrm{V}[x]$ it splits into linear factors (we work in a domain). Now given $f \in \mathrm{V}[x]$, splitting it into linear factors yields a splitting of $\overline{f} \in \mathrm{V}/\mathfrak{m}[x]$ into linear factors. Thus we see V is

Henselian, as any factorization of $\overline{f}$ breaks into these linear factors which we can lift. It is strictly Henselian since the residue field is then also algebraically closed.

Fix an AIC valuation ring V over $\operatorname{Spec}\mathrm{R}$ and $\mathfrak{p} \subset \mathrm{V}$. Any reduced quotient of a localization is also an AIC valuation ring, hence strictly Henselian. We learn that V and $\mathrm{V}/\mathfrak{p}$ are strictly Henselian valuation rings, and have identical residue fields. By standard facts about étale cohomology, we see

$$\mathrm{R}\Gamma_{\text{ét}}(\mathrm{V}) \simeq \mathrm{R}\Gamma_{\text{ét}}(\mathrm{V}/\mathfrak{p}).$$

For the same reasons, $\mathrm{R}\Gamma_{\text{ét}}(\mathrm{V}_{\mathfrak{p}}) \simeq \mathrm{R}\Gamma_{\text{ét}}(\kappa(\mathfrak{p}))$. We therefore have a cartesian square

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec}\mathrm{V}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec}\mathrm{V}/\mathfrak{p}) \arrow[d] \\
\mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec}\mathrm{V}_{\mathfrak{p}}) \arrow[r] & \mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec}\kappa(\mathfrak{p})).
\end{tikzcd}$$

It follows from Theorem 1.7 that $\mathrm{R}\Gamma_{\text{ét}}$ is an arc sheaf. □

3. Descent in mixed characteristic

In mixed characteristic one uses the arc_p topology. The appropriate definition is essentially the same, but we add some p-completeness conditions.

Definition 3.1. A ring R is derived p-complete if

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{R})}(\mathrm{S}, \mathrm{R}) = 0$$

for $\mathrm{S} \in \mathrm{D}(\mathrm{R}[1/p])$.

A ring R is also derived p-complete if and only if

$$\mathrm{Ext}^i_{\mathrm{R}}(\mathrm{R}[1/p], \mathrm{R}) = 0$$

for $i = 0$ and $i = 1$. This is generally easier to check.

If it is derived p-complete and p-adically separated, it is classically p-complete.

Example 3.2. We'd like $\mathbf{Z}_p$ to be derived p-complete. We have $(\mathbf{Z}_p)_p = \mathbf{Q}_p$. It is indeed true that $\mathrm{Hom}_{\mathrm{D}(\mathbf{Z}_p)}(\mathrm{S}, \mathbf{Z}_p) = 0$ for S a complex of $\mathbf{Q}_p$ vector spaces regarded as $\mathbf{Z}_p$-modules. More generally, a p-adically complete $\mathbf{Z}$-module R has

$$\mathrm{Hom}_{\mathrm{D}(\mathbf{Z})}(\mathrm{S}[1/p], \mathrm{R}) \simeq \lim_n \mathrm{Hom}_{\mathrm{D}(\mathbf{Z})}(\mathrm{S}[1/p], \mathrm{R}/p^n) = 0$$

via classical p-completeness.

With a more well-behaved notion of completeness for arbitrary rings, we are ready to define the arc_p topology.

> Definition 3.3. A p-arc is a rank ≤ 1 valuation ring which is p-complete and has $p \neq 0$.
>
> An arc_p-cover is a map $R \to S$ of algebras which are derived p-complete such that for every map $R \to V$ for $\operatorname{Spec} V$ a p-arc there is an extension $R \to V \to W$ of p-arcs extending over S.

This gives a notion of an arc_p-sheaf on $\mathsf{CRing}^{\mathrm{op}}_{R,p}$, where the subscript p means the ring is derived p-complete. This generalizes in the usual way to qcqs schemes which are derived p-complete.

We can also similarly define an arc_p topology on p-adic formal schemes, which is useful in prismatic cohomology.

> Definition 3.4. Consider the category fSch_p of p-adic formal schemes. We call a map $\mathfrak{Y} \to \mathfrak{X}$ of qcqs p-adic formal schemes an arc_p cover if for every map $\operatorname{Spf} V \to \mathfrak{X}$ where V is a p-arc has a faithfully flat extension to $\operatorname{Spf} W \to \mathfrak{Y}$ where W is a p-arc.

We give the category a Grothendieck topology in the usual way.

> Definition 3.5. An integral perfectoid ring R is a p-complete ring such that R/p is semiperfect, the kernel of $\theta : W(R^\flat) \to R$ is principal, and there is $\pi \in R$ so π^p is a unit multiple of p.

If R has no p-torsion, then the condition on Fontaine's map is equivalent to if $x \in R[1/p]$ and $x^p \in R$ then $x \in R$. In characteristic p, an integral perfectoid ring is just a perfect ring. We can also verify this condition by checking $(-)^p : R/\pi \to R/\pi^p$ is an isomorphism.

> Theorem 3.6. Any derived p-complete scheme X admits an arc_p-cover of the form $\operatorname{Spec}(R)$ where R is an integral perfectoid ring.

Proof. We may assume $X = \operatorname{Spec} A$. First, we explain the important idea of the construction for making a v-cover. If we take $\kappa = \max(|A|, \aleph_0)$ and let $\{A_i\}$ be a set of isomorphism class representatives of AIC valuation rings which are A-algebras of cardinality at most κ. Then $A \to \prod_i A_i$ is a v-cover, since for any morphism $A \to V$ we can enlarge V to assume it is AIC as there is an extension of valuation rings $V \to V'$ with V' AIC. Then this map $f : A \to V'$ can be factored further into $A \to V'' \to V'$ with $V'' \to V'$ an extension, and V'' AIC with cardinality at most κ: we set K to be the algebraic closure of the fraction field of $\mathrm{im}(f)$ so $|K| < \kappa$ and put $V'' = V' \cap K$. Thus, $A \to V''$ actually factors through $\prod_i A_i$ (up to an isomorphism) so we are done.

An extremely similar argument shows that if we do the same but take rank ≤ 1 valuation rings we get an arc cover; denote this by

$$A \to \prod_v A_v^+$$

where A_v^+ are all the AIC rank ≤ 1 valuation rings. Upon p-completion of each A_v^+, the same argument shows this is an arc_p cover.

It now suffices to check $\widehat{A_v^+}$ is integral perfectoid, as *arbitrary* products of integral perfectoid rings are integral perfectoid. This follows from the definition and that $A_{\mathrm{inf}}(R) = W(R^\flat)$ commutes with arbitrary products.

Consider the more general case of taking p-completion of an absolutely integrally closed domain V (every monic polynomial admits a root). If V is an $\mathbf{F}_p$-algebra, due to $x^p - \alpha$ having a root Frobenius is surjective. It is also reduced, so Frobenius cannot have any kernel and hence V is perfect.

If V is in mixed characteristic we can pick $\pi^p = p$ due to being p-torsion free and being absolutely integrally closed applied to $x^p - p$. Moreover, V/p is semiperfect by the same argument. To see $\widehat{V}_p$ is integral perfectoid the only condition left to check is that $\ker \theta$ is principal, which can be done by checking $(-)^p : V/\pi \to V/\pi^p$ is an isomorphism (this is preserved after p-completion). Due to being AIC, we can always find preimages so get surjectivity. Injectivity is automatic in this setting once we have $\pi^p | p$: if $x \pmod{\pi}$ is sent to zero under $(-)^p$, $x^p = \pi^p y$ for some $y \in V$. But then $x/\pi \in V[1/\pi]$ lies in V, as the pth power (namely y) does. It follows that $x \pmod{\pi}$ was zero. ☐

This is especially useful when working with étale cohomology, since it allows us to use arc-descent to reduce to the perfectoid setting and then we can apply tilting to reduce to charactistic p, giving us a way to move from mixed characteristic to characteristic p. We will see this technique used later in the proof of the étale comparison theorem for étale cohomology.

COROLLARY 3.7. Let $\mathscr{F}$ be a torsion étale sheaf on $\operatorname{Spec} R$, where R is derived p-complete. The functor

$$R\Gamma_{\text{ét}} : \mathsf{CRing}^{\mathrm{op}}_{R,p} \to D(\mathbf{Z})^{\geq 0}$$

sending

$$(\operatorname{Spec} S, f : \operatorname{Spec} S \to \operatorname{Spec} R) \mapsto R\Gamma(\operatorname{Spec} \widehat{S}_p[1/p]_{\text{ét}}, f^*\mathscr{F})$$

satisfies arc_p-descent.

Proof. It suffices to check descent for an arc_p-covering of rings $R \to S$. But given such a covering, we note

$$R \to S \times R/p \times R[1/p]$$

is an arc-covering. Indeed, let $\operatorname{Spec} V \to \operatorname{Spec} R$ be an arc mapping into $\operatorname{Spec} R$. We want to produce a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} W \arrow[r] \arrow[d] & \operatorname{Spec} S \times R/p \times R[1/p] \arrow[d] \\
\operatorname{Spec} V \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

where $\operatorname{Spec} W$ is an arc extending $\operatorname{Spec} V$.

The image of p under a ring map $R \to V$ is one of the following:

- A nonzero element of $\mathfrak{m}$. Then we may replace V by its p-completion without harm, and then use that $R \to S$ is an arc_p cover to find W.
- Zero, in which case $W = V$ works where we map $\operatorname{Spec} W \to \operatorname{Spec} R/p$.
- A unit, in case we do the same but with $\operatorname{Spec} R[1/p]$.

The operations of derived p-completion and inverting p kill both, hence we get an arc_p-sheaf by arc-descent of étale cohomology. ☐

4. GAGA

The arc-topology allows for a quick proof of the following result. The claim is a bit more general, but I want to highlight a useful subcase at the benefit of removing some terminology from the statement.

THEOREM 4.1 (Rigid GAGA). Let A be a Noetherian ring which is Henselian along p (e.g. p-adically complete, or p-torsion). Let X be a proper $A[1/p]$-scheme and denote X^{ad} the associated adic space over $\operatorname{Spa}(A[1/p], A)$. For any torsion étale sheaf $\mathscr{F}$ on X, write $\mathscr{F}^{\mathrm{ad}}$ for the pullback to X^{ad}. The natural map gives an isomorphism

$$R\Gamma(X, \mathscr{F}) \simeq R\Gamma(X^{\mathrm{ad}}, \mathscr{F}^{\mathrm{ad}}).$$

The main theorem we'll want to apply for this is a formal gluing result.

DEFINITION 4.2. A formal gluing datum of rings is a pair $(A \to B, I)$ where $A \to B$ is a map in CRing and $I \subset A$ is a finitely generated ideal such that $A/I^n \simeq B/I^nB$ is induced by the map $A \to B$ for all $n \geq 0$.

Note that this comes with a square

$$\begin{tikzcd} \operatorname{Spec} A & \operatorname{Spec} B \arrow[l] \\ \operatorname{Spec} A \setminus V(I) \arrow[u] & \operatorname{Spec} B \setminus V(IB). \arrow[l] \arrow[u] \end{tikzcd}$$

We will want to show that we get a cartesian square after applying an arc-sheaf to all schemes in this diagram.

THEOREM 4.3 (Formal gluing for arc-sheaves). Let $\mathscr{F} : \mathsf{Sch}^{\mathrm{op}}_{\mathrm{qcqs},R} \to D(\Lambda)^{\geq 0}$ be a finitary arc-sheaf. If $(A \to B, I)$ is a formal gluing datum of R-algebras, then we have a cartesian square

$$\begin{tikzcd} \mathscr{F}(\operatorname{Spec} A) \arrow[r] \arrow[d] & \mathscr{F}(\operatorname{Spec} B) \arrow[d] \\ \mathscr{F}(\operatorname{Spec} A \setminus V(I)) \arrow[r] & \mathscr{F}(\operatorname{Spec} B \setminus V(IB)). \end{tikzcd}$$

Sketch. We give a sketch assuming reduction to where A is an AIC valuation ring of rank ≤ 1 (this follows from showing equivalence of arc-sheaves can just be checked on such rings).

Then either A is I-adically complete or I is the unit ideal. If I is the unit ideal, the square is trivial. If A is I-adically complete, then using $A/I^nA \simeq B/I^nB$ we obtain a map

$$B \to \widehat{B} \simeq A$$

which is a section of $A \to B$.

Call the section s. Consider the diagram

$$\begin{tikzcd} \mathscr{F}(\operatorname{Spec} A) \arrow[r] \arrow[d] & \mathscr{F}(\operatorname{Spec} B) \arrow[r, "s^*"] \arrow[d] & \mathscr{F}(\operatorname{Spec} A) \arrow[d] \\ \mathscr{F}(\operatorname{Spec} A \setminus V(I)) \arrow[r] & \mathscr{F}(\operatorname{Spec} B \setminus V(IB)) \arrow[r, "s^*"] & \mathscr{F}(\operatorname{Spec} A \setminus V(I)). \end{tikzcd}$$

The right square can be shown to be homotopy cartesian. Indeed, $s : B \to A$ is surjective, and in this case as A is a valuation ring either $I = A$ in which case it's trivial or $I \subseteq \mathfrak{m}$

in which case $A/I^nA \simeq B/I^nB$ forces $A = B$ and the claim is again trivial. By the 2/3 property the left square is also homotopy cartesian, and we are done. ☐

Proof of GAGA via formal gluing. By Nagata compactification, there exists a proper A-scheme $\mathfrak{X}$ extending X. For this, we have

$$\mathfrak{X} \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,A[1/p] \simeq X.$$

Let $\tilde{\mathscr{F}}$ be a torsion étale sheaf on $\mathfrak{X}$ extending $\mathscr{F}$, and put $\widehat{\mathfrak{X}}$ for the p-adic completion of the scheme $\mathfrak{X}$. We have a cartesian square in locally ringed topoi

$$\begin{tikzcd} X^{\mathrm{ad}} \arrow[r] \arrow[d] & \widehat{\mathfrak{X}} \arrow[d] \\ X \arrow[r] & \mathfrak{X}. \end{tikzcd}$$

where we take the associated étale topology on each locally ringed space in the square.

Via pullback, $\tilde{\mathscr{F}}$ defines a sheaf on each of the topoi in the square (which we all denote by $\tilde{\mathscr{F}}$). We claim that applying $R\Gamma(-, \tilde{\mathscr{F}})$ to the square yields a cartesian square. Covering $\mathfrak{X}$ by affines, by pullback we get affine (or affinoid for X^{ad}) covers of each of $X, \mathfrak{X}, \widehat{\mathfrak{X}}, X^{\mathrm{ad}}$.

By the affinoid comparison theorem and using these covers to compute cohomology, it suffices to check the affine case. But in the affine case this follows from formal gluing for étale cohomology, using the formal gluing square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widehat{A}_p \arrow[d] \\ A[1/p] \arrow[r] & \widehat{A}_p[1/p]. \end{tikzcd}$$

Now knowing that applying $R\Gamma(-, \tilde{\mathscr{F}})$ to the square yields a cartesian square, we can deduce the main result. Note that proper base change gives an isomorphism

$$R\Gamma(\mathfrak{X}, \tilde{\mathscr{F}}) \simeq R\Gamma(\widehat{\mathfrak{X}}, \tilde{\mathscr{F}}).$$

Being a cartesian square, the claim follows. ☐