THE ALMOST PURITY THEOREM

1. Almost purity

In this setting, if K is a perfectoid field, we have the following more classical definition of a perfectoid ring:

> Definition 1.1. A perfectoid ring R over a perfectoid field K is a complete uniform Tate K-algebra R (meaning R_0 is open and bounded) and there is a pseudouniformizer (topologically nilpotent unit) $\pi \in \mathrm{R}$ such that $\pi^p | p$ and the Frobenius map
> $$\mathrm{R}^\circ/\pi \to \mathrm{R}^\circ/\pi^p$$
> is an isomorphism.

> Remark 1.2. This is completely fine with Banach K-algebras as well.

The definition does not depend on the choice of pseudouniformizer.

> Lemma 1.3. In the previous definition, we can replace the conditions after uniform Tate K-algebra with $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$. In particular, the choice of pseudouniformizer doesn't matter.

Proof. The map $\mathrm{R}^\circ/\pi \to \mathrm{R}^\circ/\pi^p$ is automatically injective when $\pi^p|p$. Given an element $x \in \mathrm{R}^\circ$, if $x \pmod{\pi}$ is sent to zero under Frobenius, $x^p = \pi^p y$ for some $y \in \mathrm{R}^\circ$. But then $x/\pi \in \mathrm{R}$ lies in R°, as the pth power (namely y) does. It follows that $x \pmod{\pi}$ was zero. Note that also that if we only assume $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$, such π exists since K is perfectoid.

Hence, the task is to show $\mathrm{R}^\circ/\pi \twoheadrightarrow \mathrm{R}^\circ/\pi^p$ if and only if $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$. One direction is easy: if we get surjectivity modulo p, then $\mathrm{R}^\circ/\pi \twoheadrightarrow \mathrm{R}^\circ/\pi^p$. Indeed, we have a square

$$\begin{array}{ccc} \mathrm{R}^\circ/p & \xrightarrow{\Phi} & \mathrm{R}^\circ/p \\ \downarrow & & \downarrow \\ \mathrm{R}^\circ/\pi & \xrightarrow{\Phi} & \mathrm{R}^\circ/\pi^p \end{array}$$

Now every map is surjective except the bottom, so the claim follows.

In the other direction, successive approximation using surjectivity of the bottom arrow shows any $x \in \mathrm{R}^\circ$ has the form

$$x = \sum_{i \geq 0} \pi^{pi} x_i^p$$

for $x_i \in \mathrm{R}^\circ$. It follows

$$x - (x_0 + \pi x_1 + \pi^2 x_2 + \ldots)^p \in p\mathrm{R}^\circ$$

by expanding. Thus, we can get any element of R°/p from Frobenius: take your desired element, find a lift x, use the above to find $x_0 + \pi x_1 + \pi^2 x_2 + \ldots$, then reduce this modulo p. □

EXAMPLE 1.4. A nice example is $\mathbf{Q}_p(p^{1/p^\infty})^\wedge\langle \mathrm{T}^{1/p^\infty}\rangle$. Here, $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ is a perfectoid field since the value group is clearly not discrete. The valuation ring is $\mathbf{Z}_p[p^{1/p^\infty}]^\wedge$, and modulo p it's semiperfect since

$$\mathbf{Z}_p[p^{1/p^\infty}]^\wedge \simeq (\mathbf{Z}_p[t^{1/p^\infty}]/(t-p))^\wedge$$

and reducing this mod p gives $\mathbf{F}_p[t^{1/p^\infty}]/t$ which is semiperfect hence Frobenius is surjective.

The Tate algebra structure is given by taking $\mathrm{R}^\circ = \mathbf{Z}_p[p^{1/p^\infty}]^\wedge\langle \mathrm{T}^{1/p^\infty}\rangle$ and picking a psuedouniformizer π.

Then R° is open and powerbounded, so it remains only to check surjectivity of Frobenius on R/p. A similar calculation shows we get another semiperfect ring.

The target theorem is the following.

THEOREM 1.5 (Almost purity). Let R be perfectoid ring. For any finite étale R-algebra S, we know S is perfectoid and the algebra S° is almost finite étale over R°.

More generally, take a perfectoid affinoid K-algebra $(\mathrm{R}, \mathrm{R}^+)$. Then $\mathrm{R}^{+a}_{\text{fét}} \simeq \mathrm{R}_{\text{fét}}$.

Let me give an example which illustrates the reason for the name.

THEOREM 1.6 (Zariski-Nagata purity). Let $\mathrm{X}/\mathbf{C}$ be a smooth scheme. For a finite morphism $f : \mathrm{Y} \to \mathrm{X}$ with Y smooth, the ramification locus of f is pure of codimension one.

The following example clarifies how to fix ramification along divisor in the local situation.

EXAMPLE 1.7. Let R be a complete regular local ring over $\mathbf{C}$, and f a regular parameter (so $\mathfrak{m} = (f)$). Then

$$\pi_1^{\text{ét}}(\operatorname{Spec} \mathrm{R} \setminus \{f = 0\}) \simeq \widehat{\mathbf{Z}},$$

with the étale cover for $\mathbf{Z}/n\mathbf{Z}$ explicitly being given by $\operatorname{Spec} \mathrm{R}[\sqrt[n]{f}] \to \operatorname{Spec} \mathrm{R} \setminus \{f = 0\}$.

The thing to have in mind is $\mathbf{C}[[x]]$, with $\mathfrak{m} = (x)$. This is geometrically a unit disk; removing 0 makes it punctured, i.e. $\mathbf{C}((x))$. The covering maps for punctured disk as the same as the circle, you look at $\mathbf{C}[[x^{1/n}]]$.

The almost purity theorem can be considered as an analogue of this theorem, which I'll explain by example (shamelessly stolen from Scholze on mathoverflow).

Consider the rings

$$\mathrm{R}_m = \mathbf{Z}_p[p^{1/p^m}, \mathrm{T}^{\pm 1/p^m}].$$

Then $\mathrm{R}_0 = \mathbf{Z}_p[\mathrm{T}^{\pm}]$ is a torus. These are all smooth over $\mathbf{Z}_p[p^{1/p^m}]$. Suppose we are handed some finite normal R_0-algebra S_0, and we know that $\mathrm{S}_0[1/p]$ is étale. The thing that prevents S_0 from being étale is that there is some possible ramification on the special fiber.

The idea is to attempt to fix this by using the ramified tower R_m, and setting $\mathrm{S}_m := \operatorname{Norm}(\mathrm{S}_0 \otimes_{\mathrm{R}_0} \mathrm{R}_m)$, where here we take the normalization of this tensor product.

This almost gets rid of the ramification when we look at $\mathrm{S}_m \to \mathrm{R}_m$. Take for example $\mathrm{S}_0 = \mathrm{R}_0[x]/(x^{2p} - p)$, and assume p is odd. Then the formal derivative $2px^{2p-1}$ is going to be invertible in

$$\mathrm{R}_0[x][1/p]/(x^{2p} - p),$$

as $2p$ is now invertible, so we just check if $2x^{2p-1}$ is invertible. But $2x^{2p-1} \cdot x = 2p$, which is again invertible in R_0. Thus, $\mathrm{R}_0 \to \mathrm{R}_0[x][1/p]/(x^{2p} - p)$ is standard étale.

Then we can note that

$$\mathrm{S}_0 \otimes_{\mathrm{R}_0} \mathrm{R}_m \simeq \mathrm{R}_m[x]/(x^{2p} - p).$$

Note that x^2 and $p^{1/p}$ both have pth powers equal to p. However, this means in the fraction field $(x^2/p^{1/p})^p = 1$, but $x^2/p^{1/p}$ is not in the ring. Thus, it is not normal.

The ring S_m will be normalized to become $\mathrm{R}_m[x]/(x^2 - p^{1/p})$, which is the same as adjoining $p^{1/(2p^m)}$. That is, we look at $\mathrm{R}_m[p^{1/(2p^m)}]$. However, this will still not be quite étale as p^{1/p^m} is not a unit.

We can measure this by looking at the ramification over the local ring of the generic point of the special fiber. Viewing the special fiber as a divisor in R_m, its generic point will have a local ring which is a DVR $\mathcal{O}_{m,s}$ with fraction field $\mathbf{Q}_p(p^{1/p^m})(\mathrm{T}^{\pm 1/p^m})$. Thus, understanding this amounts to knowing the different.

The ramification calculation amounts to understanding the ramification of $\mathcal{O}_{m,s}[p^{1/(2p^m)}]$ over $\mathcal{O}_{m,s}$. This ends up being exactly the same calculation Kush made: the different $p^{1/2p^m}$ tends towards a unit, as its valuation goes to zero. You can cook up similar examples, like $\mathrm{R}_0[x]/(x^{2p} - \mathrm{T})$, and the same sort of thing happens. Here, moving to R_m the normalization similarly forces $x^2 = \mathrm{T}^{1/p}$, as $(x/(\mathrm{T}^{1/p}))^2 = 1$. However, this case is a bit different: the same computation gives a unit for R_1!

If we suppose the ramification on the generic point of the special fiber actually becomes trivial for some m, then Zariski-Nagata purity saves the day. Indeed, the ramification locus of S_m over R_m has to be pure of codimension one, and we know what these points are. They are either characteristic zero in which case $\mathrm{S}_0[1/p]$ being étale takes care of it, or they are the generic point of the special fiber. We have just ruled out the latter, so there must be no ramification at all.

Almost purity, as stated above, tells us that it is always the case that S_∞ becomes almost étale over R_∞.

Now, how do we prove this theorem? There is an almost isomorphism $\mathrm{R}^+ \to \mathrm{R}^\circ$, so it suffices to prove it for R°. So far, we have completed the following diagram for R perfectoid:

$$\begin{array}{ccccc}
\mathrm{R}_{\text{fét}} & \longleftarrow & \mathrm{R}^{\circ a}_{\text{fét}} & \xrightarrow{\sim} & (\mathrm{R}^{\circ a}/\pi)_{\text{fét}} \\
 & & & & \big\downarrow \wr \\
\mathrm{R}^\flat_{\text{fét}} & \underset{\sim}{\longleftarrow} & \mathrm{R}^{\flat\circ a}_{\text{fét}} & \xrightarrow{\sim} & (\mathrm{R}^{\flat\circ a}/\pi^\flat)_{\text{fét}}
\end{array}$$

where the arrow on the bottom left is because we know almost purity in characteristic p. What remains is almost purity, or rather showing the top morphism is actually an isomorphism.

From tilting, we know already $\mathrm{R}^\flat_{\text{fét}} \to \mathrm{R}_{\text{fét}}$ is fully faithful. Also, we know the theorem in the case of a field. Thus, the real content of the theorem is that the functor is actually essentially surjective in general outside of the case of a field.

The idea of the proof is to use the geometry of the perfectoid space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ to localize and reduce to the case of a field, where we already know the full story.

This crucially uses rigid analytic geometry: if we were to try the same strategy with $\mathrm{Spec}\,\mathrm{R}$, the stalks would be far from correct since we need to get an actual field after completion. There are not enough opens in the Zariski topology to accomplish this.

For this, we will need some definitions in rigid analytic geometry.

2. Perfectoid spaces: crash course

The formalism for perfectoid spaces I will use here is Huber's adic spaces, because this is largely what has been adopted in many papers that use these objects.

Let K be a non-archimedean field. The idea of rigid analytic spaces or more generally adic spaces is to try to emulate what happens with complex analytic spaces over $\mathbf{C}$ in a non-archimedean setting. There are several models of rigid-analytic geometry one can use. I'll be using adic spaces since Scholze uses these for perfectoid spaces.

On important requirement is that there should be an analytification functor

$$\mathrm{Var}_K \to \{\text{rigid analytic spaces}/K\}$$

and we should also expect some form of GAGA to hold when the variety is proper. Another important thing is that unlike algebraic geometry, for $f \in \Gamma(\mathrm{X}^{\mathrm{ad}}, \mathcal{O}_{\mathrm{X}^{\mathrm{ad}}})$ we should be able to make sense of not just the vanishing locus of f but also the set $\{x \in \mathrm{X}^{\mathrm{ad}} : |f(x)| \leq 1\}$. As we'll see, this is essentially the added content that makes it analytic when compared to algebraic geometry.

This idea means that we ought to be able to attach, for any $x \in \mathrm{X}^{\mathrm{ad}}$, a valuation function

$$f \mapsto |f(x)|$$

on functions. Let me be precise about what valuation means - these are really more like seminorms, but this is the usual terminology.

Definition 2.1. Let R be a ring. A valuation on R is a multiplicative map

$$|\cdot| : \mathrm{R} \to \Gamma \cup 0$$

where Γ is a totally ordered abelian group (written multiplicatively). We ask that $|0| = 0, |1| = 1$ and also

$$|x + y| \leq \max(|x|, |y|)$$

for all $x, y \in \mathrm{R}$.

We say two valuations are equivalent if for all $a, b \in \mathrm{R}$ we have $|a| \geq |b|$ if and only if $|a|' \geq |b|'$.

If R has a topology, which will be the case in our situation, we ask that $\{x \in \mathrm{R} : |x| < \gamma\}$ is always open.

Our next task is to define the equivalent of affine schemes in the rigid analytic world, affinoids. In the adic space formalism, these are specified by a pair of rings $(\mathrm{R}, \mathrm{R}^+)$ to which we associate a space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ of certain continuous valuations on R.

We will restrict ourselves to the case of algebras over a non-archimedean field, as the full generality of Huber's theory isn't needed here.

DEFINITION 2.2. A Tate K-algebra R is a topological K-algebra R for which there exists a subring $\mathrm{R}_0 \subset \mathrm{R}$ such that $a\mathrm{R}_0$ for $a \in K^\times$ forms a basis of open neighborhoods of 0.

This comes with a ring R° that consists of powerbounded elements. This means R° consists of $x \in \mathrm{R}$ such that $\{x^n : n \geq 0\} \subset a\mathrm{R}_0$ for some a.

An affinoid K-algebra is a pair $(\mathrm{R}, \mathrm{R}^+)$ consisting of a Tate K-algebra R and an open integrally closed subring $\mathrm{R}^+ \subset \mathrm{R}^\circ$. A morphism of affinoid K-algebras is a K-algebra map $\mathrm{R} \to \mathrm{S}$ carrying R^+ to S^+.

The typical example takes R to be a quotient of $K\langle \mathrm{T}_i \rangle$, and $\mathrm{R}^+ = \mathrm{R}^\circ = K^\circ\langle \mathrm{T}_i \rangle$. In everything that follows, $|f(x)|$ denotes the valuation x applied to the function $f \in \mathrm{R}$. We use $[x]$ to denote the equivalence class of a valuation.

DEFINITION 2.3. Given an affinoid K-algebra $(\mathrm{R}, \mathrm{R}^+)$, we give $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+) := \{[x] : |f(x)| \leq 1, f \in \mathrm{R}^+\}$ the topology with basis given by the open rational subsets

$$\mathrm{U}\left(\frac{f_1, \ldots, f_n}{g}\right) := \{x \in \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+) : |f_i(x)| \leq |g(x)| \neq 0\}.$$

Here, f_i generate the unit ideal. Essentially, we ask that $|g(x)| \neq 0$ is open (like $D(g)$ in algebraic geometry) and $|f(x)| \leq 1$ is open (a feature we want in rigid geometry).

This topological space only depends on the completion of R. **Hence, from now on we will assume $(\mathrm{R}, \mathrm{R}^+)$ is complete.**

We will eventually want to make this the underlying topological space in an upgraded notion of a locally ringed space; the role is analogous to Spec R in algebraic geometry.

The pair $(\mathrm{R}, \mathrm{R}^+)$ should be thought of as R providing functions to $(\mathbf{A}^1_K)^{\mathrm{ad}}$, and R^+ providing functions to the adic unit disk $\mathrm{Spa}(K\langle \mathrm{T} \rangle, K^\circ\langle \mathrm{T} \rangle)$ (in particular, it's functions which are ≤ 1 and hence 'summable').

This is literally true once the correct definitions of these are in place.

I want to first demystify the integrally closed condition on R^+: it actually doesn't make any change in generality.

LEMMA 2.4. There is a bijection between sets of equivalence classes $F_S = \bigcap_{f \in S} \{[x] : |f(x)| \leq 1\}$ for non-empty subsets $S \subset \mathrm{R}^\circ$ and open and integrally closed subrings $\mathrm{R}^+ \subseteq \mathrm{R}^\circ$.

We send

$$\mathrm{R}^+ \mapsto \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+) = \{[x] : |f(x)| \leq 1, f \in \mathrm{R}^+\},$$

and

$$F_S \mapsto \{f \in \mathrm{R} : |f(x)| \leq 1 \text{ for all } x \in F_S\}.$$

The conditions on R^+ (being an open and integrally closed) arise naturally as a result of this lemma.

Thus, we can think of the set $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ as imposing the condition $|f(x)| \leq 1$ for f in a particular subset on the space of valuations. Taking $\mathrm{R}^+ = \mathrm{R}^\circ$ means we impose the most conditions. It is not reasonable to ask for such a bound when f is not power bounded. In summary, R^+ is really just encoding which functions $f \in \mathrm{R}$ are ≤ 1, or the functions to the unit disk.

I also want to explain why it is necessary to allow possibilities for R^+ other than R°. Indeed, if you look at classical rigid geometry we only use this. One important reason is that even when defining $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^\circ)$ we would want the open rational subsets to again be affinoids.

Definition 2.5. Let $\mathrm{U}\left(\frac{f_1,\ldots,f_n}{g}\right)$ be a rational open in $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$. Let

$$\mathrm{B} \subseteq \mathrm{R}\left[\frac{f_i}{g}\right] \subseteq \mathrm{R}[g^{-1}]$$

be the integral closure of $\mathrm{R}^+[\frac{f_i}{g}]$ in $\mathrm{R}[\frac{f_i}{g}]$. Topologizing $\mathrm{R}[\frac{f_i}{g}]$ by making $a\mathrm{R}_0[\frac{f_i}{g}]$ for $a \in K^\times$ a basis of opens at 0, we get an affinoid K-algebra $(\mathrm{R}[\frac{f_i}{g}], \mathrm{B})$. Upon completion, we get $(\mathrm{R}\langle\frac{f_i}{g}\rangle, \widehat{\mathrm{B}})$.

This pair has a universal property that shows it depends only on U: that for a map $(\mathrm{R}, \mathrm{R}^+) \to (\mathrm{S}, \mathrm{S}^+)$ factoring over U with $(\mathrm{S}, \mathrm{S}^+)$ complete there is a unique map

$$\left(\mathrm{R}\left\langle\frac{f_i}{g}\right\rangle, \widehat{\mathrm{B}}\right) \to (\mathrm{S}, \mathrm{S}^+)$$

making the obvious diagram

$$\begin{array}{ccc} \left(\mathrm{R}\left\langle\frac{f_i}{g}\right\rangle, \widehat{\mathrm{B}}\right) & \longrightarrow & (\mathrm{S}, \mathrm{S}^+) \\ & \nwarrow & \uparrow \\ & & (\mathrm{R}, \mathrm{R}^+) \end{array}$$

commute.

Note that completion preserves the adic spectrum and gives better behaved rings which is why we performed this operation.

Having noted the independence from U, define presheaves $\mathcal{O}_X(U), \mathcal{O}_X^+(U)$ on $X = \mathrm{Spa}(R, R^+)$ by

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left(R\left\langle \frac{f_i}{g} \right\rangle, \widehat{B}\right)$$

on rational opens, and for general opens by taking a limit:

$$\mathcal{O}_X(W) = \lim_{U \subseteq W \text{ rational}} \mathcal{O}_X(U),$$

for example. We note that these are not always sheaves, but are in good situations e.g. topologically of finite type.

Now for some standard facts:

> THEOREM 2.6. For a rational open, we have $U \simeq \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. The stalks $\mathcal{O}_{X,x}$ are local rings, and furthermore for *any* open U we have
> $$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1, x \in U\}.$$

This first fact is not surprising given the universal property. Note that it shows why R^+ is needed: if we want rational opens to be affinoids, we can't just use R°.

We have now equipped equipped X with a locally ringed space structure! To account for the sheaf $\mathcal{O}_X^+$, we make a modification.

> DEFINITION 2.7. The category $\mathcal{V}$ consists of locally ringed topological spaces $(X, \mathcal{O}_X)$ where X is a sheaf of complete topological K-algebras along with a continuous valuation
> $$f \mapsto |f(x)|$$
> on $\mathcal{O}_{X,x}$ for every $x \in X$.
>
> A morphism is a morphism of locally topologically ringed spaces which are continuous K-algebra morphisms on $\mathcal{O}_X$ and compatible with the valuations.

The data of $\mathcal{O}_X^+$ is given by the valuations: we always have

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1, x \in U\}.$$

Thus, (R, R^+) sometimes naturally gives an object in $\mathcal{V}$, which we call affinoid adic spaces. The tricky thing is that $\mathcal{O}_X$ need not be a sheaf; call such a pair *sheafy*.

DEFINITION 2.8. An adic space is an object of $\mathcal{V}$ locally isomorphic to an sheafy affinoid $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$.

REMARK 2.9. If R is topologically finite type, then $(\mathrm{R}, \mathrm{R}^+)$ is sheafy. This is not surprising, since these correspond to rigid analytic spaces (there is a fully faithful functor from rigid analytic spaces with topologically finite type adic spaces as the essential image).

Being stably uniform also shows a pair is sheafy: this means all rational opens are uniform affinoids, or S° is bounded for a rational subdomain $\mathrm{Spa}(\mathrm{S}, \mathrm{S}^+) \subset \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$.

After this formalism, it's important go through some examples. One of the standard examples is the adic unit disk, which illustrates why we chose to allow valuations of rank > 1.

EXAMPLE 2.10 (The disk). Consider the adic space $\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle)$.

Let me point out some obvious points of this space. For any $x \in \mathbf{C}_p$ with $|x| \leq 1$, we obtain a valuation

$$f \mapsto |f(x)|_{\mathbf{C}_p} \in \mathrm{R}_{>0}$$

by literally evaluating $f \in \mathbf{C}_p\langle \mathrm{T}\rangle$ at x. This corresponds to a maximal ideal of $\mathbf{C}_p\langle \mathrm{T}\rangle$, by taking the kernel of $f \mapsto f(x) \in \mathbf{C}_p$. These classical points are why this is called the (closed) adic unit disk.

Next, we can see the use of having valuations which are rank > 1. You might attempt to decompose this as a topological space via

$$\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle) = \{x : |\mathrm{T}(x)| = 1\} \cup \{x : \cup_{\varepsilon>0} |\mathrm{T}(x)| < 1 - \varepsilon\}$$

which are both open. Geometrically, this is breaking a closed disk into the open disk and the boundary.

If we only had these classical points, as in rigid analytic geometry, the disk would fail to be connected (this is part of why there we cannot use an honest topology). However, points corresponding to rank > 1 valuations fix the issue. In general there are 5 types of points, the 5th one having value group $\mathrm{R}_{>0} \times \gamma^{\mathbf{Z}}$ with lexicographic ordering and $\gamma > 1$. For $x \in \mathbf{C}_p$ with $|x| \leq 1$ and $r \in (0, 1]$, define when $f = \sum_{n \geq 0} a_n (\mathrm{T} - x)^n$ the valuation

$$|f(x_{r^-})| = \max_n |a_n| r^n \gamma^{-n} \in \mathrm{R}_{>0} \times \gamma^{\mathbf{Z}}.$$

There is also x_{r^+}, where we take positive powers of γ.

> This indeed gives a valuation, and we will note that 0_{1^-} does not lie in either open. Putting $f = \mathrm{T}$, $|\mathrm{T}(0_{1^-})| = (1, \gamma^{-1})$. This does not equal 1, but it is also not $\leq 1 - \varepsilon$ due to the rank two ordering.

Note that if you take $K = \mathbf{C}_p$, the functor of points of $\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle)$ spits out R^+ (which justifies the earlier slogan).

Finishing the proof of almost purity requires a stalk calculation in adic spaces. It is here that we see why Spec R really fails as this task.

Let $k(x)$ be the residue field of $\mathcal{O}_{\mathrm{X},x}$, and $k(x)^+$ the image of $\mathcal{O}^+_{\mathrm{X},x}$.

> LEMMA 2.11. Let X be an adic space over K, and suppose $\pi \in K$ is atopologically nilpotent unit. Then for $x \in \mathrm{X}$ the π-adic completion of $\mathcal{O}^+_{\mathrm{X},x}$ is the π-adic completion of $k(x)^+$.

Proof. This is local, so assume $\mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$. There is a surjective map $\mathcal{O}^+_{\mathrm{X},x} \to k(x)^+$, with kernel I. Letting $f \in \mathrm{I}$, note that $\pi^{-1} f \in \mathrm{I}$. Indeed, $\mathrm{V} = \{[x] : |f(x)| \leq |\pi(x)|\}$ is a valid open set. But then $\mathcal{O}^+_{\mathrm{X}}(\mathrm{V}) \supseteq \mathrm{R}^+[f/\pi]$ if $\mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$, so $f/\pi \in \mathcal{O}^+_{\mathrm{X},x}$. It must be in the kernel since f is, so I is π-divisible. Therefore, after π-adic completion we get an isomorphism. □

This behavior is not present in schemes, since such open sets do not exist and so will not be part of the stalk. Having understood this surprising point about stalks, it's now more believable that we can deduce almost purity from the field case by looking at stalks.

There are also examples of adic spaces which are not topologically finite type. The relevant class of adic spaces for us will be perfectoid spaces.

> DEFINITION 2.12. Let K be a perfectoid field. A perfectoid space X is an adic space which is locally isomorphic to an affinoid perfectoid space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$, which simply means R is a perfectoid ring over K.

Note that R^+ is almost isomorphic to R°: the ring $\mathrm{R}^\circ/\mathrm{R}^+$ is almost zero, since if $x \in \mathrm{R}^\circ$ then πx is topologically nilpotent and as R^+ is open we see $(\pi x)^n \in \mathrm{R}^+$, which means $\pi x \in \mathrm{R}^+$ by being integrally closed.

Contained in this definition is a nontrivial theorem that the structure sheaf is a sheaf, so that perfectoid spaces lie in $\mathcal{V}$. This follows from the following important theorem, which is where the hard part of the proof is done.

THEOREM 2.13. Let R be a perfectoid K-algebra. Then for a rational subdomain $\mathrm{U} \subseteq \mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ the K-algebra $\mathcal{O}_{\mathrm{X}}(\mathrm{U})$ is perfectoid.

Sketch. First, if we are working in characteristic p the claim is easier to prove but still nontrivial.

If $\mathrm{U} = \mathrm{U}\left(\frac{f_1,\ldots,f_n}{g}\right)$ then $\mathcal{O}_{\mathrm{X}}(\mathrm{U})$ has as a ring of definition the completion of $\mathrm{R}^+\left[\frac{f_i}{g}\right]$. Thus, we get $\mathcal{O}_{\mathrm{X}}(\mathrm{U})$ after inverting π in $\widehat{\mathrm{R}^+\left[\frac{f_i}{g}\right]}$. One may append $f_{n+1} = \pi^N$ for some large N without changing U, so without loss of generality put $f_n = \pi^N$. Indeed, we know $(f_1, \ldots, f_n) = 1$ so there are h_i so $\sum_i h_i f_i = 1$. Then for $N \gg 0$, $\pi^N h_i \in \mathrm{R}^+$ as it is open. Then $|\pi^N(x)| = |(\sum_i \pi^N h_i f_i)(x)| \leq \max_i |\pi^N h_i(x)||f_i(x)| \leq |g(x)|$. By scaling, we may assume f_i and g are in R^+.

Since we work in characteristic p and everything is perfect, there are inclusions

$$\mathrm{R}^+\left[\frac{f_i}{g}\right] \subseteq \mathrm{R}^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \subseteq \mathrm{R}[1/g].$$

Call the first inclusion θ. One can show that $\mathrm{coker}(\theta)$ is killed by a power of π^{nN}, so inverting π in $\mathrm{R}^+\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\rangle$ gives the same Tate K-algebra. But this is clearly perfectoid in this presentation, so $\mathcal{O}_{\mathrm{X}}(\mathrm{U})$ is perfectoid.

Indeed, note that it suffices to show $\pi^{nN}\prod_i(\frac{f}{g})^{1/p^{n_i}}$ lies in $\mathrm{R}^+[f_n/g] \subseteq \mathrm{R}^+[f_i/g]$. But this product just becomes $\prod_i \frac{f_n}{g}(f_i^{1/p^{n_i}} g^{1-1/p^{n_i}})$, which shows the claim.

Next, leverage the claim in characteristic p to prove the claim in mixed characteristic for rational subdomains of the form

$$\mathrm{U} = \mathrm{U}\left(\frac{f_1,\ldots,f_n}{g}\right)$$

where $f_i = a_i^\sharp$ and $g = b^\sharp$ for $a_i, b \in \mathrm{R}^\flat$. One can show, through essentially the exact same argument, that there is again an explicit description

$$\mathcal{O}_{\mathrm{X}}(\mathrm{U}) \simeq \mathrm{R}^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle[1/\pi]$$

which shows it is perfectoid.

Then, since $\sharp$ is not surjective, one must use a technical approximation lemma for elements of R in terms of elements of $\mathrm{R}^\flat$ to show all rational subsets can be written this way. $\square$

COROLLARY 2.14. If X is a perfectoid space, then for $x \in X$ we have $\widehat{\mathcal{O}^+_{X,x}} \simeq \widehat{k(x)^+}$ where $\widehat{k(x)}$ is a perfectoid field.

Proof. We have

$$\widehat{k(x)^+} = \mathrm{colim}_{x \in U} \widehat{\mathcal{O}_X}(U)^+$$

over rational subsets U containing x. But then $\mathcal{O}_X(U)^+$ is perfectoid over $K^{\circ a}$, and a completed filtered colimit of perfectoid $K^{\circ a}$-algebras is again perfectoid. Thus, $\widehat{k(x)^+}$ is perfectoid, and $\widehat{k(x)}$ is a perfectoid field. □

The previous corollary be important in reducing the proof of almost purity to the case of a field. Importantly, knowing $\mathcal{O}_X(U)$ is perfectoid also allows us to deduce that perfectoid spaces are adic spaces.

COROLLARY 2.15. Let R be a perfectoid ring, and set $X = \mathrm{Spa}(R, R^+)$. Then $\mathcal{O}_X$ is a sheaf.

Proof. The previous theorem implies that for any affinoid U the ring $\mathcal{O}_X(U)^\circ$ is bounded as $\mathcal{O}_X(U)$ is perfectoid. This implies that X is stably uniform, which implies that (R, R^+) is sheafy. □

3. PROOF OF ALMOST PURITY

Now we're ready to use this new geometry to help localize the proof of almost purity to the case of fields, following Scholze's argument.

DEFINITION 3.1. A map $f : (A, A^+) \to (B, B^+)$ of affinoid K-algebras is finite étale if $A \to B$ is, and B^+ is the integral closure of $f(A^+)$ in B.

This extends to adic spaces in the obvious way by taking a cover by affinoids.

DEFINITION 3.2. Suppose we are working with perfectoid K-algebras. Then a map is *strongly finite étale* if additionally B^+ is almost finite étale over A^+.

Recall a map $A \to B$ of almost $K^{\circ a}$-algebras being finite étale means almost finitely presented, almost flat, and unramified which means there is a diagonal idempotent $e \in (B \otimes_A B)_*$. Specifically, $e^2 = e, \ker(\mu_*) \cdot e = 0, \mu_*(e) = 1$ where μ is the multiplication map $B \otimes_A B \to B$.

> This also globalizes to perfectoid spaces in the obvious way.

This definition will later be redundant after we know almost purity.

The following lemma is nontrivial, but I omit the proof since it is a reasonable statement.

> LEMMA 3.3. If Y is affinoid perfectoid and $f : X \to Y$ is strongly finite étale, then X is affinoid perfectoid and also
> $$(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$$
> is strongly finite étale.

The following is now basically immediate from the lemma.

> COROLLARY 3.4. Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space. The following are true:
>
> - We have $\mathrm{sf\acute{e}t}(X) \simeq R^{+a}_{\mathrm{f\acute{e}t}}$. In particular, we can tilt the category of strongly finite étale maps.
> - The functor $\mathrm{sf\acute{e}t}(X) \to R_{\mathrm{f\acute{e}t}}$ is fully faithful.
> - For rational subsets $U \subseteq X$, the assignment
> $$U \mapsto \mathrm{sf\acute{e}t}(U)$$
> is a sheaf of categories.

Proof. For the first assertion, the previous lemma shows that we can check strongly finite étale on global sections in the affinoid case (which a priori needs to be checked on a cover by affinoids). In particular, strongly finite étale maps are the same as almost finite étale covers of R^+ (the condition on $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ being étale is automatic from $\mathcal{O}_Y^+(Y) \to \mathcal{O}_X^+(X)$ being almost finite étale). It is tiltable because after the equivalence we can work in the almost setting, where this is already done.

For the second assertion, after we have the first claim this reduces to the full faithfulness we already knew in almost purity.

For the final assertion, this is again precisely the previous lemma. □

The strategy is now to show that there is an equivalence of sheaves of categories
$$\eta : \mathrm{sf\acute{e}t}(U) \to \mathcal{O}_X(U)_{\mathrm{f\acute{e}t}}$$
given by inverting π. Here, we view these as sheaves on rational opens of the adic space X.

Knowing η is fully faithful for each individual U and that sfét(U) is a sheaf of categories on K-rational subsets of X, we are then reduced to verifying that as a presheaf $\mathcal{O}_X(U)_{\text{fét}}$ is separated and that η is an equivalence on stalks.

Here, $\mathcal{F}$ separated means that

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. The main content is checking an equivalence on stalks, which we will do by reducing to almost purity for fields.

For computing stalks, we will need some nontrivial input from Gabber-Romero's book.

> Theorem 3.5 (Elkik, Gabber-Romero). Let R be a flat K°-algebra Henselian along (π). Then
>
> $$R[1/\pi]_{\text{fét}} \simeq \widehat{R}[1/\pi]_{\text{fét}}$$
>
> where on the right we have π-adically completed R.

> Corollary 3.6. Let (A_i, A_i^+) be a filtered system of complete uniform affinoid K-algebras. Note that we genuinely need the generality of filtered systems for applications, and not just **N**-indexed. In elementary terms, C being filtered means the system is non-empty and for every $c_1, c_2 \in C$ there is c_3 and morphisms $c_1, c_2 \to c_3$. Additionally, if $f, g : c_1 \to c_2$ are parallel, there is a morphism $h : c_2 \to c_3$ so $hf = hg$.
>
> Set $(A, A^+) := (A^+[1/\pi], \widehat{\operatorname{colim} A_i^+})$. Then
>
> $$2 - \operatorname{colim}(A_i)_{\text{fét}} \simeq A_{\text{fét}}.$$

Proof. The idea is just that we have equivalences

$$\begin{array}{ccc} \operatorname{colim}_i (A_i)_{\text{fét}} & & A_{\text{fét}} \\ \downarrow \sim & & \| \\ (\operatorname{colim}_i (A_i^+)[1/\pi])_{\text{fét}} & \xrightarrow{\sim} & A^+[1/\pi]_{\text{fét}} \end{array}$$

The downward left arrow is an isomorphism by definition.

Note that the resulting ring $\operatorname{colim}_i(A_i^+)$ is still Henselian along π, since filtered colimits preserve being Henselian. Hence, the Elkik-Gabber-Romero result applies for the bottom arrow. The final equality is by definition. □

We will apply this corollary when computing stalks of the sheaf $\mathcal{O}_X(U)_{\text{fét}}$. Now we will use the outlined strategy to reduce to a stalk computation to prove almost purity.

> THEOREM 3.7 (Almost purity). The morphism η is an equivalence of sheaves of categories on rational subsets of $X = \mathrm{Spa}(R, R^+)$.

Proof. As discussed, it suffices to check the equivalence on stalks. Precisely, for any $x \in X$ we must show

$$\mathrm{colim}_{x\in U}\, \mathcal{O}_X^+(U)^a_{\text{fét}} \simeq \mathrm{colim}_{x\in U}\, \mathcal{O}_X(U)_{\text{fét}}$$

via inverting π.

Via tilting and almost purity in characteristic p,

$$\mathrm{colim}_{x\in U}\, \mathcal{O}_X^+(U)^a_{\text{fét}} \simeq \mathrm{colim}_{x^\flat\in U^\flat}\, \mathcal{O}_{X^\flat}^+(U^\flat)^a_{\text{fét}} \simeq \mathrm{colim}_{x^\flat\in U^\flat}\, \mathcal{O}_{X^\flat}(U^\flat)_{\text{fét}}.$$

Now observe that the stalk

$$\mathrm{colim}_{x^\flat\in U^\flat}\, \mathcal{O}_{X^\flat}^+(U^\flat)$$

is Henselian along $\pi^\flat$, by the fact that being Henselian is preserved by filtered colimits. Moreover, it has $\pi^\flat$-adic completion equal to $\widehat{k(x^\flat)^+}$, by Lemma 2.11.

Then it follows by Corollary 3.6 that

$$\mathrm{colim}_{x^\flat\in U^\flat}\, \mathcal{O}_{X^\flat}(U^\flat)_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}.$$

Therefore, we see $\mathrm{colim}_{x\in U}\, \mathcal{O}_X^+(U)^a_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}$.

Similarly, we can also do the untilted version for $\mathrm{colim}_{x\in U}\, \mathcal{O}_X(U)_{\text{fét}}$, which gives $\widehat{k(x)}_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}$ via tilting for perfectoid fields.

Tracing through these equivalences, since almost purity in characteristic p inverts $\pi^\flat$ the overall equivalence is given by inverting π. $\square$

This completes the proof of almost purity after taking global sections.