Rigid flat connections and p-curvature

May 30, 2023

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Let $X/\mathbf{C}$ be a smooth projective variety.

Theorem (Simpson)

There exists a quasi-projective moduli space $\mathcal{M}_{\mathrm{dR}}(X, \mathcal{L}, r)$ *of stable rank* r *flat vector bundles* (E, ∇) *together with* $\det E \simeq (\mathcal{L}, \nabla_{\mathcal{L}})$*.*

We say (E, ∇) is *rigid* if $[(E, \nabla)]$ is an isolated point in the moduli scheme. We will make the additional assumption that $\mathcal{L}$ is torsion throughout.

Simpson's motivicity conjecture

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Conjecture (Motivicity)

Any rigid flat connection is a subquotient of a Gauss-Manin connection.

This is asking that the underlying local system of flat sections is a summand of $\mathbf{R}^i f_* \mathbf{C}$, where $f : Y \to U \subseteq X$ is smooth projective and U is a dense open.

Curvature

In characteristic 0, connections on E are flat/integrable if and only if the induced map

$$\mathrm{Der}_{\mathrm{X}/k} \to \mathrm{End}_k(\mathrm{E})$$

via sending $\partial \mapsto \nabla_\partial$ is a Lie algebra homomorphism, defined as the composition

$$\mathrm{E} \xrightarrow{\nabla} \mathrm{E} \otimes \Omega^1_{\mathrm{X}/k} \xrightarrow{\partial} \mathrm{E} \otimes \mathcal{O}_{\mathrm{X}} \simeq \mathrm{E}.$$

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The usual notion is $K = \nabla^2 : E \to E \otimes \Omega^2_{X/k}$ being zero. Our map being a Lie algebra homomorphism asks for

$$[\nabla_{\partial_1}, \nabla_{\partial_2}] - \nabla_{[\partial_1, \partial_2]} = 0,$$

but this is actually the composition

$$E \xrightarrow{K} E \otimes \Omega^2_{X/k} \xrightarrow{\partial_1 \wedge \partial_2} E \otimes \mathcal{O}_X \simeq E.$$

Thus, this also characterizes flat connections. We will see this is a more fruitful perspective.

In characteristic p, flat connections can have *p-curvature*. This is when

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fails to be a map of sheaves of restricted Lie algebras.

A *restricted Lie algebra* is Lie algebra $\mathfrak{g}$ over a field k of characteristic $p > 0$, equipped with an additional p-operation $\mathrm{X} \to \mathrm{X}^{[p]}$. This must satisfy some additional axioms.

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Example

Let G/k be a reductive group. Then $\mathfrak{g}$ is a restricted Lie algebra with the p-operation induced by Frobenius.

One motivation for this, beyond simply observing extra Lie algebra structure exists, is that vanishing curvature and p-curvature ensures E is spanned Zariski-locally by solutions to $\nabla(e) = 0$.

p-curvature

For a flat connection ∇, p-curvature is a map

$$\psi : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathrm{X}/k}$$

given by

$$\psi(e)(\partial) := \nabla^p_\partial(e) - \nabla_{\partial^p}(e).$$

For any derivation ∂ this defines an endomorphism of E. We obtain a map

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vanishing iff the map induced by the connection is a homomorphism of restricted Lie algebras.

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vanishing iff the map induced by the connection is a homomorphism of restricted Lie algebras.
The following is due to Katz, which connects this to the motivicity conjecture.

Theorem (Katz)

Let k be a perfect field of characteristic $p > 0$. Then Gauss-Manin connections have nilpotent p-curvature.

Main theorem

Subquotients of the Gauss-Manin connections have nilpotent p-curvature, so a consequence of motivicity is that mod p reductions of rigid flat connections have nilpotent p-curvature when p is large.

Theorem (Esnault-Groechenig)

Let X *be a smooth connected projective variety over* $\mathbf{C}$*, and let* (E, ∇) *be a rigid flat connection with torsion determinant* $\mathcal{L}$*. Then there is a scheme* $\mathrm{S}/\operatorname{Spec}\mathbf{Z}$ *of finite type so that* $(\mathrm{X}, (\mathrm{E}, \nabla))$ *has a model* $(\mathrm{X}_\mathrm{S}, (\mathrm{E}_\mathrm{S}, \nabla_\mathrm{S}))$ *such that for every closed point* s (E_s, ∇_s) *has nilpotent* p*-curvature.*

This is proved without assuming motivicity, and so provides some evidence for the conjecture.
Unless otherwise noted, all results are due to Esnault and Groechenig.

Main theorem

The Higgs field of a stable rigid Higgs bundle is always nilpotent, otherwise we can produce a nontrivial deformation with $\lambda\theta$: this can be seen by looking at the effect on the Hitchin map

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For flat connections we cannot use this simple idea in characteristic p as $\lambda\nabla$ need not be a connection.

Since it works over $\mathbf{C}$ we can get nilpotence on the Higgs side even for reductions X_s. In non-abelian Hodge theory in characteristic p, the corresponding condition is **nilpotence of** p**-curvature**. Thus, we attempt to use non-abelian Hodge theory in characteristic p to leverage this and prove the theorem.

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More precisely, the proof idea is the following:

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- Note that we can make rigid Higgs bundles automatically nilpotent, like they are over $\mathbf{C}$.
- Using Ogus-Vologodsky's results on non-abelian Hodge theory in characteristic p, use $\mathrm{char}(k(s)) > \mathrm{D}$ to make rigid Higgs bundles on $\mathrm{X}_s^{(p)}$ sufficiently nilpotent to correspond to rigid flat vector bundles on X_s with nilpotent p-curvature.

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- Construct a model $(\mathrm{X}_{\mathrm{S}}, \mathcal{L}_{\mathrm{S}})$ with good properties. We want models for rigid flat connections and rigid Higgs bundles so the statement makes sense.
- Note that we can make rigid Higgs bundles automatically nilpotent, like they are over $\mathbf{C}$.
- Using Ogus-Vologodsky's results on non-abelian Hodge theory in characteristic p, use $\mathrm{char}(k(s)) > \mathrm{D}$ to make rigid Higgs bundles on $\mathrm{X}_s^{(p)}$ sufficiently nilpotent to correspond to rigid flat vector bundles on X_s with nilpotent p-curvature.
- Deduce the result by comparing the number of rigid Higgs bundles on $\mathrm{X}_s^{(p)}$ and rigid flat vector bundles on X_s. This uses usual non-abelian Hodge theory over $\mathbf{C}$.

Arithmetic models

Let $(\mathrm{X}, \mathcal{L})$ be as before.

Definition

A arithmetic model is a morphism

$$\mathrm{X}_{\mathrm{S}} \to \mathrm{S}$$

along with a line bundle $\mathcal{L}_{\mathrm{S}}$ such that:

- S is finite type and smooth over $\operatorname{Spec} \mathbf{Z}$.
- S has a unique generic point η, with $k(\eta) \subseteq \mathbf{C}$.
- Along the map $\operatorname{Spec} \mathbf{C} \to \mathrm{S}$, the base change map induces an isomorphism

$$\mathrm{X}_{\mathrm{S}} \times_{\mathrm{S}} \operatorname{Spec} \mathbf{C} \simeq \mathrm{X}.$$

- The map $\mathrm{X}_{\mathrm{S}} \to \mathrm{S}$ is smooth and projective.
- There is $\mathcal{L}_{\mathrm{S}} \in \operatorname{Pic}(\mathrm{X}_{\mathrm{S}})$ pulling back to $\mathcal{L}$.

Nice models

W want to choose a particularly nice arithmetic model that gives good properties for the moduli space. There are moduli schemes

$$\mathcal{M}_{\mathrm{dR}}(\mathrm{X_S}/\mathrm{S}, \mathcal{L}_{\mathrm{S}}, r) \to \mathrm{S}$$

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Definition

Given a moduli scheme $\mathcal{M} \to \mathrm{S}$, let $\mathcal{M}^{\mathrm{rig}}$ be the largest open subscheme where the structure map is quasifinite.

We want certain arithmetic models we call *nice* models is to ensure good properties of these moduli spaces with respect to rigidity.

Nice models

For every positive integer r it is shown there exists an affine arithmetic scheme S and an arithmetic model $(\mathrm{X}_{\mathrm{S}}, \mathcal{L}_{\mathrm{S}})$ of $(\mathrm{X}, \mathcal{L})$ with several good properties.

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(Spreading out). For every rigid flat connection $(\mathrm{E}_{\mathbf{C}}, \nabla_{\mathbf{C}})$ over X with determinant $\mathcal{L}$ and rank $\leq r$, there exists a spreading out $(\mathrm{E_S}, \nabla_\mathrm{S})$ over $\mathrm{X_S}$ which is stable at geometric points. The same holds for rigid Higgs bundles, *and* θ_S *can be assumed nilpotent.*

We want this property to be able to see information about Higgs bundles and flat connections over **C** so that we have models for (E, ∇) like in the target theorem.

Next, we want some compatibility with rigidity: understanding the rigid connections on X_s is important in the strategy.

(Compatibility with the rigid locus I). We ask that the S points

$$[E_S, \nabla_S] : S \to \mathcal{M}_{dR}(X_S/S, \mathcal{L}_S, \leq r)$$

and also $[E_S, \theta_S]$ given by the previous property factor through the subschemes $\mathcal{M}^{rig}_{dR}(X_S/S, \mathcal{L}_S, \leq r)$ and $\mathcal{M}^{rig}_{Dol}(X_S/S, \mathcal{L}_S, \leq r)$.

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(Compatibility with the rigid locus II). Further, for every

$$y \in |\mathcal{M}^{rig}_{dR}(X_S/S, \mathcal{L}_S, \leq r)|,$$

there exists a family (X_S, ∇_S) such that $y \in [E_S, \nabla_S](|S|)$ (the set-theoretic image). The same holds for Higgs bundles.

Frobenius twists

Before moving to the next step of the proof let us review some definitions about Frobenius twists.

- $\mathrm{X}^{(p)} := \mathrm{X} \times_f \operatorname{Spec} k$ where $f : k \to k$ sends $x \mapsto x^p$.

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- The absolute Frobenius F is given on rings by $x \mapsto x^p$, giving an k-scheme X a map

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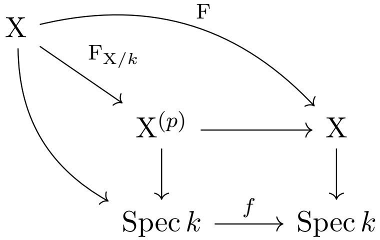
- The relative Frobenius $\mathrm{F}_{\mathrm{X}/k}$ is defined by the diagram



Theorem (Ogus-Vologodsky)

Let k be a perfect field of characteristic $p > 0$. Then a lifting $\mathcal{X} \to \operatorname{Spec} \mathrm{W}_2(k)$ of $\mathrm{X} \to \operatorname{Spec} k$ induces an equivalence of categories

$$C^{-1}_{\mathcal{X}/\mathrm{W}_2(k)} : \mathsf{Higgs}_{p-1}(\mathrm{X}^{(p)}) \to \mathsf{MIC}_{p-1}(\mathrm{X}).$$

The left hand side is the category of stable Higgs bundles with nilpotence $\leq p-1$ ($\theta^{p-1} = 0$), and the right side is flat vector bundles (E, ∇) with degree of nilpotence of p-curvature $\leq p-1$ ($\psi(\nabla)^{p-1} = 0$).

This will allow us to transport nilpotence on the Higgs side to our desired nilpotence of p-curvature.

Let $F_{X/k}$ be the relative Frobenius $X \to X^{(p)}$, given by universality of the fibre product. Under this,

$$C^{-1}(E, 0) = (F_{X/k}^* E, \nabla^{\mathrm{can}}).$$

The canonical connection is uniquely characterized by

$$\nabla^{\mathrm{can}}(s) = 0$$

if and only if $s \in F_{X/k}^{-1} E(U)$ (it is naturally in $F_{X/k}^* E(U)$).

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if and only if $s \in F^{-1}_{X/k}E(U)$ (it is naturally in $F^*_{X/k}E(U)$). The Cartier transform C^{-1} can also be used to extend the classical result of Deligne-Illusie:

$$(F_{X/k})_* \tau_{<p-\ell}(E \otimes \Omega^\bullet_{X/k}) \simeq \tau_{<p-\ell}(C_{\mathcal{X}/W_2(k)}(E) \otimes \Omega^\bullet_{X^{(p)}/k})$$

if E is nilpotent of level ℓ. Take $E = (\mathcal{O}_X, d)$ to recover the result: $C^{-1}(\mathcal{O}_{X^{(p)}}, 0)$ yields this.

Applying Ogus-Vologodsky

Pick a nice model X_S. The following uses the compatibility with the rigid locus.

Proposition (Esnault-Groechenig)

There exists an integer $D > 0$ depending on X_S/S such that for any closed $s \in S$ with $\mathrm{char}(k(s)) > D$ the Ogus-Vologodsky correspondence sends any rigid stable Higgs bundle (V_s, θ_s) to a rigid stable connection $C^{-1}(V_s, \theta_s)$.

The next point is to observe that the output here has nilpotent p-curvature, so it allows us to produce what we hope to be all rigid stable connections out of Higgs bundles.

Remark

Crucially, when picking a nice model we can assume θ_S is nilpotent. Control over the residue characteristic allows us to control the degree of nilpotence, so Ogus-Vologodsky can apply.

Counting connections

Let $n_{\mathrm{dR}}(\mathrm{X}, \mathcal{L}, r)$ to be the number of stable rigid flat connections of rank r on X with determinant $\mathcal{L}$. Similarly define $n_{\mathrm{Dol}}(\mathrm{X}, \mathcal{L}, r)$. Let $n_{\mathrm{dR}}^{\mathrm{nilp}}(\mathrm{X}, \mathcal{L}, r)$ denote the number which are nilpotent.

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We want $n_{\mathrm{dR}}^{\mathrm{nilp}}(\mathrm{X}, \mathcal{L}, r) \geq n_{\mathrm{dR}}(\mathrm{X}, \mathcal{L}, r)$. A corollary of the previous result is the following, after taking a nice model X_{S}.

Corollary

We have

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To see this, apply C^{-1} to each item of the RHS. In the Ogus-Vologodsky correspondence, it lands in MIC_{p-1} and hence has nilpotent p-curvature. Because we have an equivalence of categories, it must be an injection.

Counting connections

From non-abelian Hodge theory, we'd expect $n_{\mathrm{Dol}}(\mathrm{X}_s^{(p)}, \mathcal{L}_s^{(p)}, r)$ to be closely related to $n_{\mathrm{dR}}(\mathrm{X}_s, \mathcal{L}_s, r)$.

Lemma

We have $n_{\mathrm{Dol}}(\mathrm{X}_s^{(p)}, \mathcal{L}_s^{(p)}, r) = n_{\mathrm{dR}}(\mathrm{X}_s, \mathcal{L}_s, r)$.

Proof.

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- For closed points $s \in \mathrm{S}$, given we work with a nice model X_{S} both are equal to their respective numbers over $\mathbf{C}$.
- We have $n_{\mathrm{Dol}}(\mathrm{X}, \mathcal{L}, r) = n_{\mathrm{dR}}(\mathrm{X}, \mathcal{L}, r)$ by non-abelian Hodge theory.

□

Putting everything together

To summarize, what we now know is the following when the residue characteristic of s is $> \mathrm{D}$:

- Using Ogus-Vologodsky and the theorem telling us how it behaves on rigid stable Higgs bundles, we get

$$n_{\mathrm{dR}}(\mathrm{X}_s, \mathcal{L}_s, r) \geq n_{\mathrm{dR}}^{\mathrm{nilp}}(\mathrm{X}_s, \mathcal{L}_s, r) \geq n_{\mathrm{Dol}}(\mathrm{X}_s^{(p)}, \mathcal{L}_s^{(p)}, r).$$

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- Further, $n_{\mathrm{Dol}}(\mathrm{X}_s^{(p)}, \mathcal{L}_s^{(p)}, r) = n_{\mathrm{dR}}(\mathrm{X}_s, \mathcal{L}_s, r)$.

The claimed result then follows for sufficiently large residue characteristic. The most difficult content lies in the construction of nice models, and also in using these to show C^{-1} preserves rigidity.