F-GAUGES AND CRYSTALLINE GALOIS REPRESENTATIONS

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CONTENTS

1. THE STACK X^{Syn}

At this point, we've defined a stack

$$t : X^{Nyg} \to \mathbf{A}^1/\mathbf{G}_m$$

which sees all of the cohomology theories we want except for étale cohomology.

The starting point is to recall the statement of the étale comparison.

THEOREM 1.1. Let $X = \operatorname{Spf} R$ be an affine p-adic formal scheme over a perfectoid ring A/d corresponding to a perfect prism (A, d). Then

$$R\Gamma_{\text{ét}}(X_\eta, \mathbf{Z}_p) \simeq \Delta_{X/A}[1/d]^{\varphi=1}.$$

The idea of the proof is to use descent to reduce to the case where X is perfectoid, where after tilting the claim follows from

$$R\Gamma_{\text{ét}}(\operatorname{Spec} R^\flat, \mathbf{Z}_p) \simeq \operatorname{fib}(\varphi - 1 : W(R^\flat) \to W(R^\flat)),$$

which is due to the Artin-Schreier sequence.

In fact, there is a better version with coefficients on perfectoids.

THEOREM 1.2. Let $X = \operatorname{Spf} R$ be a perfectoid ring with corresponding to a perfect prism (A, d). Then

$$R\Gamma_{\text{ét}}(X_\eta, \mathbf{Z}_p(n)) \simeq [\ \varphi^{-1}(d^n)A \xrightarrow{\varphi/d^n - 1} A\ ]$$

The fact that $\varphi^{-1}(d^n)$ appears immediately lets us realize that if we want to do this with general coefficients, we are going to need Nygaard filtered prismatic cohomology.

To properly do this, we need Breuil-Kisin twists.

DEFINITION 1.3. The Breuil-Kisin twist $\mathcal{O}\{1\}$ on $\mathbf{Z}_p^{\mathrm{Nyg}}$ is defined as

$$\pi^*\mathcal{O}_{\Delta}\{1\} \otimes t^*\mathcal{O}(-1)$$

where $\mathcal{O}(-1)$ is the usual line bundle on $\mathbf{A}^1/\mathbf{G}_m$.

Note that $\mathcal{O}_{\Delta}\{-1\}$ can be constructed geometrically, as $\mathcal{H}^2_{\Delta}(\mathbf{P}^1_{\mathbf{Z}_p})$. Otherwise, the idea is to lift $\mathrm{I}_{\Delta}/\mathrm{I}_{\Delta}^2$ on the Hodge-Tate stack to a prismatic crystal. On a transversal prism, if we put

$$\mathrm{I}_r = \mathrm{I} \otimes_{\mathrm{A}} \varphi^*\mathrm{I} \otimes \varphi^{2*}\mathrm{I} \otimes \ldots \otimes \varphi^{(r-1)*}\mathrm{I}$$

then $\mathcal{O}_{\Delta}\{1\}(\mathrm{A}, \mathrm{I})$ is given by the limit of

$$\cdots \twoheadrightarrow \mathrm{I}_3/\mathrm{I}_3^2 \twoheadrightarrow \mathrm{I}_2/\mathrm{I}_2^2 \twoheadrightarrow \mathrm{I}/\mathrm{I}^2.$$

In particular, it lifts I/I^2 and $\varphi^*\mathrm{A}\{1\} \simeq \mathrm{I}^{-1} \otimes \mathrm{A}\{1\}$.

This defines $\mathcal{O}\{n\}$ in general on $\mathrm{X}^{\mathrm{Nyg}}$ via pullback. We now consider the perfectoid case.

PROPOSITION 1.4. Assume R is perfectoid, and pick a trivialization $\mathbf{A}_{\mathrm{inf}}(\mathrm{R}^\flat)\{1\} \simeq \mathbf{A}_{\mathrm{inf}}(\mathrm{R}^\flat)$ and pick a generator d of I. Then

$$j_{\mathrm{HT}}^* : \mathrm{R}\Gamma(\mathrm{R}^{\mathrm{Nyg}}, \mathcal{O}\{n\}) \to \mathrm{R}\Gamma(\mathrm{X}^{\Delta}, \mathcal{O}\{n\})$$

identifies with

$$\varphi\{n\} = \varphi/d^n : \mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_{\mathrm{R}} \to \Delta_{\mathrm{R}}$$

and j_{dR}^* identifies with

$$\mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_{\mathrm{R}} \hookrightarrow \Delta_{\mathrm{R}}.$$

REMARK 1.5. It is important here that really $j_{\mathrm{HT}}^*\mathcal{O}\{1\} = j_{\mathrm{dR}}^*\mathcal{O}\{1\} = \mathcal{O}_{\Delta}\{1\}$ so that the maps have the correct target. Clearly $j_{\mathrm{dR}}^*\mathcal{O}\{1\} = \mathcal{O}_{\Delta}\{1\}$: $\pi \circ j_{\mathrm{dR}} = \mathrm{id}$, and $t \circ j_{\mathrm{dR}}$ factors over $\mathbf{G}_m/\mathbf{G}_m$ so $\mathcal{O}(-1)$ trivializes. For j_{HT}^*, we get

$$F^*\mathcal{O}^{\Delta}_{\mathbf{Z}_p}\{1\} \otimes j_{\mathrm{HT}}^* t^*\mathcal{O}(-1).$$

The latter factor becomes I_{Δ}, and the first becomes $\mathrm{I}_{\Delta}^{-1} \otimes \mathcal{O}_{\Delta}\{1\}$ (think about the heuristic definition). Thus we get the correct result.

In particular, for a perfectoid X we get

$$\mathrm{R}\Gamma_{\text{ét}}(\mathrm{X}_\eta, \mathbf{Z}_p(n)) \simeq \mathrm{fib}(\varphi\{n\} - 1 : \mathrm{R}\Gamma(\mathrm{X}^{\mathrm{Nyg}}, \mathcal{O}\{n\}) \to \mathrm{R}\Gamma(\mathrm{X}^{\Delta}, \mathcal{O}\{n\}).$$

Thus in general, there is a map

$$\mathrm{fib}(\varphi\{n\} - 1 : \mathrm{R}\Gamma(\mathrm{X}^{\mathrm{Nyg}}, \mathcal{O}\{n\}) \to \mathrm{R}\Gamma(\mathrm{X}^{\mathbb{\Delta}}, \mathcal{O}\{n\}) \to \mathrm{R}\Gamma_{\text{ét}}(\mathrm{X}_\eta, \mathbf{Z}_p(n))$$

by arc descent of étale cohomology. The left side is not an arc sheaf, so this is not an equivalence.

These maps really arose as j_{HT}^* and j_{dR}^*. However, it is also quite important that we have a canonical identification between j_{HT}^* and j_{dR}^* in order for what we did above to make any sense.

We then introduce a new stack to make this identification, called $\mathrm{X}^{\mathrm{Syn}}$.

Definition 1.6. The stack $\mathrm{X}^{\mathrm{Syn}}$ is defined as the pushout diagram

$$\begin{array}{ccc} \mathrm{X}^{\mathbb{\Delta}} \sqcup \mathrm{X}^{\mathbb{\Delta}} & \xrightarrow{j_{\mathrm{HT}} \sqcup j_{\mathrm{dR}}} & \mathrm{X}^{\mathrm{Nyg}} \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow \\ \mathrm{X}^{\mathbb{\Delta}} & \xrightarrow{j_{\mathbb{\Delta}}} & \mathrm{X}^{\mathrm{Syn}} \end{array}$$

All horizontal maps are immersions, so this is also a pullback diagram.

Lemma 1.7. We have

$$\mathsf{QCoh}(\mathrm{X}^{\mathrm{Syn}}) \simeq \mathrm{eq}\Big(\mathsf{QCoh}(\mathrm{X}^{\mathrm{Nyg}}) \underset{j_{\mathrm{dR}}^*}{\overset{j_{\mathrm{HT}}^*}{\rightrightarrows}} \mathsf{QCoh}(\mathrm{X}^{\mathbb{\Delta}}) \Big),$$

and the analogous statement for cohomology also holds.

Proof. After applying QCoh we obtain a pullback diagram

$$\begin{array}{ccc} \mathsf{QCoh}(\mathrm{X}^{\mathrm{Syn}}) & \xrightarrow{j_{\mathbb{\Delta}}^*} & \mathsf{QCoh}(\mathrm{X}^{\mathbb{\Delta}}) \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathsf{QCoh}(\mathrm{X}^{\mathrm{Nyg}}) & \xrightarrow{j_{\mathrm{HT}}^* \sqcup j_{\mathrm{dR}}^*} & \mathsf{QCoh}(\mathrm{X}^{\mathbb{\Delta}}) \times \mathsf{QCoh}(\mathrm{X}^{\mathbb{\Delta}}) \end{array}$$

from which it follows formally that $\mathsf{QCoh}(\mathrm{X}^{\mathrm{Syn}})$ is the equalizer (this is the general categorical recipe for getting an equalizer out of a product and a pullback square).

The same holds for cohomology. □

As a result of this description, note that we get $\mathcal{O}\{1\}$ on the syntomification. If R is perfectoid, we can describe $\mathsf{QCoh}(\mathrm{R}^{\mathrm{Syn}})$ very explicitly.

EXAMPLE 1.8. Let R be a perfectoid ring, and let $(\Delta_{\mathrm{R}}, \mathrm{I})$ be the corresponding perfect prism. Picking a generator d, we have seen

$$\mathrm{R}^{\mathrm{Nyg}} = \mathrm{Spf}\, \Delta_{\mathrm{R}}\langle u, t\rangle/(ut - \varphi^{-1}(d))/\mathbf{G}_m.$$

From this description one can think of an object in $\mathsf{QCoh}(\mathrm{R}^{\mathrm{Nyg}})$ as a diagram of Δ_{R}-complexes

$$\cdots \underset{t}{\overset{u}{\rightleftarrows}} M^{i+1} \underset{t}{\overset{u}{\rightleftarrows}} M^{i} \underset{t}{\overset{u}{\rightleftarrows}} M^{i-1} \underset{t}{\overset{u}{\rightleftarrows}} \cdots$$

where $ut = tu = \varphi^{-1}(d)$: this is simply interpreting sheaves on this stack as graded modules over $\Delta_{\mathrm{R}}\langle u, t\rangle/(ut - \varphi^{-1}(d))$.

Put $M^{-\infty}$ for the completed colimit along the t maps and M^{∞} for the completed colimit along the u maps. If M is a perfect complex, the maps $M^0 \to M^{\pm\infty}$ are $\varphi^{-1}(d)$-isogenies.

We have $j_{\mathrm{dR}}^* M = M^{-\infty}$ and $j_{\mathrm{HT}}^* M = \varphi^* M^{\infty}$. Thus objects in $\mathsf{QCoh}(\mathrm{R}^{\mathrm{Syn}})$ can be described the same way, except that we add an identification

$$\tau : \varphi^* M^{\infty} \simeq M^{-\infty}.$$

It is useful to also see how $\mathsf{QCoh}(\mathrm{R}^{\mathrm{Syn}})$ can be described for a qrsp ring.

PROPOSITION 1.9. Let R be a qrsp ring. Then the category $\mathsf{QCoh}(\mathrm{R}^{\mathrm{Syn}})$ is equivalent to the category of triples

$$(M, \mathrm{Fil}^{\bullet} M, \tilde{\varphi}_M)$$

where $M \in \mathrm{D}^{\wedge}_{(p,\mathrm{I})}(\Delta_{\mathrm{R}})$, $\mathrm{Fil}^{\bullet} M \in \mathrm{DF}^{\wedge}_{(p,\mathrm{I})}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}})$ is a filtration on M, and

$$\tilde{\varphi}_M : \mathrm{Fil}^{\bullet} M \to \mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}} \otimes M$$

is linear over the filtered Frobenius $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}} \to \mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}}$ and induces an isomorphism

$$\varphi_M : \mathrm{Fil}^{\bullet} M \otimes_{\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}}} \mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}} \simeq \mathrm{I}^{\bullet} M$$

in $\mathrm{DF}^{\wedge}_{(p,\mathrm{I})}(\mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}}) \simeq \mathrm{D}^{\wedge}_{(p,\mathrm{I})}(\Delta_{\mathrm{R}})$.

Proof. Recall that we showed that $\mathrm{R}^{\mathrm{Nyg}} = \mathcal{R}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}})$. This means sheaves can be viewed $\mathrm{DF}^{\wedge}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}})$.

The de Rham open immersion is given by the cartesian square

$$\begin{array}{ccc} \mathrm{R}^{\Delta} & \longrightarrow & \mathbf{G}_m/\mathbf{G}_m \\ \downarrow{\scriptstyle j_{\mathrm{dR}}} & & \downarrow \\ \mathcal{R}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}}) & \xrightarrow{t} & \mathbf{A}^1/\mathbf{G}_m. \end{array}$$

The Hodge-Tate open immersion is given by taking the filtered Frobenius

$$\bigoplus_{i\in\mathbf{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_{\mathrm{R}} t^{-i} \to \bigoplus_{i\in\mathbf{Z}} \mathrm{I}^i\Delta_{\mathrm{R}} t^{-i},$$

which induces

$$j_{\mathrm{HT}} : \mathrm{R}^{\Delta} \simeq \mathrm{Spf}\left(\bigoplus_{i\in\mathbf{Z}} \mathrm{I}^i\Delta_{\mathrm{R}} t^{-i}\right)/\mathbf{G}_m \to \mathcal{R}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}}).$$

On sheaves, if we view them as objects in $\mathrm{DF}^{\wedge}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}})$, it is also possible to write down j^*_{dR} and j^*_{HT} explicitly. We use the equivalence

$$\mathrm{DF}^{\wedge}_{(p,\mathrm{I})}(\mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}}) \simeq \mathrm{D}^{\wedge}_{(p,\mathrm{I})}(\Delta_{\mathrm{R}}),$$

given by the equivalence of stacks we used to define j_{HT}.

Under this, the de Rham pullback of $\mathrm{Fil}^{\bullet}M$ gets sent to $\mathrm{I}^{\mathbf{Z}}M$ (after using the equivalence $\mathrm{R}^{\Delta} \simeq \mathrm{Spf}\left(\bigoplus_{i\in\mathbf{Z}} \mathrm{I}^i\Delta_{\mathrm{R}} t^{-i}\right)/\mathbf{G}_m$) and the Hodge-Tate pullback is $\mathrm{Fil}^{\bullet}M \otimes_{\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}\Delta_{\mathrm{R}}} \mathrm{I}^{\mathbf{Z}}\Delta_{\mathrm{R}}$.

We can now directly give functors in both directions. Given $M \in \mathsf{QCoh}(\mathrm{R}^{\mathrm{Syn}})$, we send

$$M \mapsto (M|_{\mathrm{R}^{\Delta}}, M|_{\mathrm{R}^{\mathrm{Nyg}}}, \tau)$$

where τ is the identification between the de Rham and Hodge-Tate pullbacks.

Conversely, $\mathrm{Fil}^{\bullet}M$ reconstructs the gauge $\mathcal{E}|_{\mathrm{R}^{\mathrm{Nyg}}}$, and φ_M gives the desired data to upgrade this to an F-gauge. □

2. Etale realization

There are two main ways to construct the étale realization. I will focus on the method that passes through prismatic F-crystals in perfect complexes.

> Lemma 2.1 (Remark 6.3.4). Restriction to X^{Δ} gives a functor
> $$\mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \to \mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}).$$

Proof. The perfectness of the restriction is clear, so the additional claim is that the resulting perfect complex $\mathcal{E}$ on X^{Δ} comes equipped with a natural identification

$$\varphi^*\mathcal{E}[1/\mathrm{I}_{\Delta}] \simeq \mathcal{E}[1/\mathrm{I}_{\Delta}].$$

Consider the maps

$$\varphi^* j_{\mathrm{dR}}^*(\mathcal{E}) \longleftarrow \varphi^*\pi_*\mathcal{E} \longrightarrow j_{\mathrm{HT}}^*(\mathcal{E})$$

associated to $\mathcal{E} \in \mathsf{Perf}(\mathrm{X}^{\mathrm{Nyg}})$. The left map is defined by applying φ^* to the canonical map

$$\pi_*\mathcal{E} \to j_{\mathrm{dR}}^*(\mathcal{E}).$$

This is obtained from viewing j_{dR} as a map of stacks over X^{Δ}, i.e. that it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{X}^{\Delta} & \xrightarrow{\;j_{\mathrm{dR}}\;} & \mathrm{X}^{\mathrm{Nyg}} \\ & {\scriptstyle \mathrm{id}}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathrm{X}^{\Delta} & \end{array}$$

The right map is obtained via adjunction to the analogous map $\varphi^*\pi_*\mathcal{E} \to j_{\mathrm{HT}}^*(\mathcal{E})$. This comes from the analogous diagram

$$\begin{array}{ccc} \mathrm{X}^{\Delta} & \xrightarrow{\;j_{\mathrm{HT}}\;} & \mathrm{X}^{\mathrm{Nyg}} \\ & {\scriptstyle \varphi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathrm{X}^{\Delta} & \end{array}$$

Now we use adjunction to see that

$$\mathrm{Hom}(\varphi^*\pi_*\mathcal{E}, j_{\mathrm{HT}}^*(\mathcal{E})) \simeq \mathrm{Hom}(\pi_*\mathcal{E}, \varphi_* j_{\mathrm{HT}}^*(\mathcal{E})).$$

But since $(\pi \circ j_{\mathrm{HT}}) = \varphi$, this is the same as a map

$$\pi_*\mathcal{E} \to \pi_*(j_{\mathrm{HT}})_* j_{\mathrm{HT}}^*\mathcal{E}.$$

There is always a canonical map

$$\mathcal{E} \to (j_{\mathrm{HT}})_* j_{\mathrm{HT}}^*\mathcal{E}$$

via the unit of the adjunction.

If we show these are I_{Δ}-isogenies we are done, since the F-gauge structure identifies $j_{\mathrm{dR}}^*(\mathcal{E})$ and $j_{\mathrm{HT}}^*(\mathcal{E})$.

By quasisyntomic descent of perfect complexes, we can check this key claim locally on quasiregular semiperfectoid rings R. The idea is then that if $\mathcal{E}$ is perfect it has Hodge-Tate weights in $[a, b]$ where $a \geq b$, and then one can show that the map φ_M (using the description of F-gauges on a qrsp) has image contained in $\mathrm{I}^a M$ and the image contains $\mathrm{I}^b M$. This implies that it has finite height. □

We denote this functor by

$$(-)|_{\mathrm{X}^{\Delta}} : \mathsf{Perf}(\mathrm{X}^{\mathrm{Syn}}) \to \mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}).$$

There is then an obvious functor

$$\mathsf{Perf}^{\varphi}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}) \to \mathsf{Perf}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathrm{I}_{\Delta}]^{\wedge})^{\varphi=1}.$$

We will see the target category is equivalent to the category of derived lisse sheaves on X_η. To give the equivalence, we first note the following results:

PROPOSITION 2.2. Let $\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ be the category of bounded complexes in $\mathrm{D}_{\text{ét}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ which are derived p-complete and each cohomology sheaf is lisse (after reduction mod p).

Then $\mathrm{X} \mapsto \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p)$ is a quasisyntomic sheaf, and for R qrsp we have

$$\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spf}\,\mathrm{R}_\eta, \mathbf{Z}_p) \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spf}\,\mathrm{R}^{\mathrm{perf}}_\eta, \mathbf{Z}_p)$$

where $\mathrm{R}^{\mathrm{perf}}$ is the perfectoidization. Thus

$$\mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spf}\,\mathrm{X}_\eta, \mathbf{Z}_p) \simeq \lim_{\mathrm{Spf}\,\mathrm{R}\to\mathrm{X}} \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spf}\,\mathrm{R}_\eta, \mathbf{Z}_p)$$

over perfectoid rings R.

REMARK 2.3.

The following result builds off of Katz's Riemann-Hilbert correspondence.

THEOREM 2.4 (Katz). Let $\mathrm{Spec}\,\mathrm{R}/\,\mathrm{Spec}\,\mathbf{F}_p$ be a scheme. Then

$$\mathrm{D}^{b}_{\mathrm{lisse}}(\mathrm{Spec}\,\mathrm{R}, \mathbf{Z}_p) \simeq \mathsf{Perf}(\mathrm{W}(\mathrm{R}))^{\varphi=1}.$$

The equivalence sends $\mathbb{L} \mapsto \mathbb{L} \otimes \mathrm{W}(\mathrm{R})$, and the inverse is taking φ fixed points.

THEOREM 2.5 (Bhatt-Scholze). Let X be a quasisyntomic p-adic formal scheme. Then

$$\mathsf{Perf}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathrm{I}_{\Delta}]^{\wedge})^{\varphi=1} \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_\eta, \mathbf{Z}_p).$$

Sketch. The essential idea is that if X is perfectoid, then the claim is

$$\mathsf{Perf}(\Delta_{\mathrm{R}}[1/\mathrm{I}_{\Delta}]^{\wedge})^{\varphi=1} \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}^{\flat}_{\eta}, \mathbf{Z}_p) \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}_{\eta}, \mathbf{Z}_p).$$

The second equivalence is tilting. The first equivalence follows from the fact that R corresponds to a perfect prism $(\mathrm{W}(\mathrm{R}^{\flat}), (d))$ where $\mathrm{R}^{\flat}$ is perfect so

$$\Delta_{\mathrm{R}}[1/\mathrm{I}_{\Delta}]^{\wedge} = \mathrm{W}(\mathrm{R}^{\flat}[1/d])$$

and a theorem of Katz says

$$\mathsf{Perf}(\mathrm{W}(\mathrm{R}^{\flat}[1/d]^{\wedge}))^{\varphi=1} \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{Spec}\,\mathrm{R}^{\flat}[1/d], \mathbf{Z}_p) \simeq \mathrm{D}^{(b)}_{\mathrm{lisse}}(\mathrm{X}^{\flat}_{\eta}, \mathbf{Z}_p).$$

Recall that for an affinoid adic space $\mathrm{Spa}(\mathrm{A}, \mathrm{A}^{+})$ that finite étale covers are the same as finite étale covers of A.

One can reduce the general claim to the perfectoid case. To make the reduction for $\mathsf{Perf}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathrm{I}_{\Delta}]^{\wedge})^{\varphi=1}$, the main claim is that for a prism (A, I) we have

$$\mathsf{Perf}(\mathrm{A}[1/\mathrm{I}]^{\wedge})^{\varphi=1} \simeq \mathsf{Perf}(\mathrm{A}_{\mathrm{perf}}[1/\mathrm{I}]^{\wedge})^{\varphi=1}.$$

Once this is shown

$$\mathsf{Perf}(\mathrm{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathrm{I}_{\Delta}]^{\wedge})^{\varphi=1} \simeq \lim_{(\mathrm{A}_{\mathrm{perf}}, \mathrm{I})} \mathsf{Perf}(\mathrm{A}_{\mathrm{perf}}[1/\mathrm{I}]^{\wedge})^{\varphi=1}$$

over perfect prisms $(\mathrm{A}_{\mathrm{perf}}, \mathrm{I})$ with a map $\mathrm{Spf}\,\mathrm{A}_{\mathrm{perf}}/\mathrm{I} \to \mathrm{X}$. Equivalently, we take the limit over perfectoid rings mapping to X, so by the previous proposition the claim follows.

By derived p-completeness, you can check this modulo p. We can assume the prism has I principal, and in this case the claim is that for an $\mathbf{F}_p$-algebra B an element t for which B is derived t-complete that

$$\mathsf{Perf}(\mathrm{B}[1/t])^{\varphi=1} \simeq \mathsf{Perf}(\mathrm{B}_{\mathrm{perf}}[1/t])^{\varphi=1} \simeq \mathsf{Perf}(\mathrm{B}^{\wedge}_{\mathrm{perf}}[1/t])^{\varphi=1}$$

where the completion is t-adic. The first is an equivalence by Katz's Riemann-Hilbert equivalence and topological invariance of the étale site. The second functor is fully faithful as perfect complexes are automatically derived t-complete, and it is essentially surjective due to Katz's Riemann-Hilbert equivalence and Elkik approximation, which says lisse sheaves on $\mathrm{Spec}\,\mathrm{B}^{\wedge}_{\mathrm{perf}}[1/t]$ are pulled back from lisse sheaves on $\mathrm{B}_{\mathrm{perf}}[1/t]$. □

This produces an étale realization functor as desired.

3. Crystalline Galois representations

We will now introduce reflexive sheaves on $\mathcal{O}_K^{\mathrm{Syn}}$.

Proposition 3.1. Let $(\mathrm{A}, \mathrm{I}) = (\mathrm{W}(k)[[x]], \mathrm{E}(x))$ be a Breuil-Kisin prism for $\mathcal{O}_K$. Then the map

$$\pi_{\mathrm{W}(k)[[u]]} : \mathcal{R}(\mathrm{I}^{\bullet}\mathrm{A}) \to \mathcal{O}_K^{\mathrm{Nyg}}$$

> is locally surjective in the flat topology.
>
> Here, we (p, I)-adically complete the Rees algebra.

Proof. Let (A, I) denote the Breuil-Kisin prism we chose. Then picking a generator $d = (u - p)$, we have

$$\mathrm{Rees}(\mathrm{I}^\bullet \mathrm{A}) = \frac{\mathrm{W}(k)[[x]]\langle u, t\rangle}{ut - \mathrm{E}(x)}$$

and the proposed stack is the quotient by $\mathbf{G}_m$ where t has degree one and u has degree -1.

The functor of points of $\mathcal{R}(\mathrm{I}^\bullet \mathrm{A})$ assigns to a p-nilpotent ring S a map $f : \mathrm{A} \to \mathrm{S}$ killing some power of I, a line bundle $L \in \mathrm{Pic}(\mathrm{S})$, and a factorization

$$\mathrm{I} \otimes_{\mathrm{A}} \mathrm{S} \xrightarrow{u} L \xrightarrow{t} \mathrm{S}$$

of the canonical map.

We get the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{I} \otimes_{\mathrm{A}} \mathbf{G}_a^{\#} & \longrightarrow & \mathrm{I} \otimes_{\mathrm{A}} \mathrm{W} & \longrightarrow & \mathrm{I} \otimes_{\mathrm{A}} F_* \mathrm{W} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle u^{\#}} & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \mathrm{V}(L)^{\#} & \longrightarrow & M_u & \longrightarrow & \mathrm{I} \otimes_{\mathrm{A}} F_* \mathrm{W} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle t^{\#}} & & \big\downarrow{\scriptstyle d_{u,t}} & & \big\downarrow & & \\
0 & \longrightarrow & \mathbf{G}_a^{\#} & \longrightarrow & \mathrm{W} & \xrightarrow{F} & F_* \mathrm{W} & \longrightarrow & 0
\end{array}$$

of W-modules over S where the composite downward maps are the canonical maps.

This diagram gives a point of $\mathbf{Z}_p^{\mathrm{Nyg}}(\mathrm{S})$, and the middle column gives a map

$$\mathcal{O}_K = \mathrm{A}/\mathrm{I} \to \mathrm{R}\Gamma(\mathrm{Spf}\,\mathrm{S}, \mathrm{W}/M_u)$$

so this further gives a map

$$\mathcal{R}(\mathrm{I}^\bullet \mathrm{A}) \to \mathcal{O}_K^{\mathrm{Nyg}}.$$

We now claim this map is a flat cover. Let A_∞ denote the perfection of the Breuil-Kisin prism. Then

$$\mathrm{A} \to \mathrm{A}_\infty$$

is a (p, I)-complete quasisyntomic cover, and hence so is $\mathrm{R} \to \mathrm{R}_\infty$. Then we have a diagram

$$\begin{array}{ccc} \mathcal{R}(\mathrm{I}^\bullet \mathrm{A}_\infty) & \xrightarrow{\sim} & \mathrm{R}^{\mathrm{Nyg}}_\infty \\ \downarrow & & \downarrow \\ \mathcal{R}(\mathrm{I}^\bullet \mathrm{A}) & \longrightarrow & \mathrm{R}^{\mathrm{Nyg}} \end{array}$$

where the vertical maps are fpqc covers. In particular, the map is a local isomorphism the fpqc topology. □

As a consequence of this proposition, it makes sense to make the following definition.

Definition 3.2. A coherent sheaf on $\mathcal{O}_K^{\mathrm{Syn}}$ is a sheaf whose pullback to $\mathcal{R}(\mathrm{I}^\bullet\mathrm{A})$ is coherent for a Breuil-Kisin prism (A, I) corresponding to $\mathcal{O}_K$.

We also now recall the notion of a reflexive module from algebraic geometry.

Definition 3.3. Let A Noetherian regular integral domain. A module $M \in \mathsf{Coh}(\mathrm{A})$ is *reflexive* if the canonical map

$$M \to M^{\vee\vee}$$

is an isomorphism.

Following this definition, we define a reflexive sheaf on $\mathcal{O}_K^{\mathrm{Syn}}$ as follows:

Definition 3.4. A reflexive sheaf on $\mathcal{O}_K^{\mathrm{Syn}}$ is a sheaf in $\mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ so that after pullback along the cover

$$\mathrm{Spf}\ \mathrm{Rees}(\mathrm{I}^\bullet\mathrm{A})/\mathbf{G}_m \to \mathcal{O}_K^{\mathrm{Nyg}} \to \mathcal{O}_K^{\mathrm{Syn}}$$

the resulting graded module is reflexive as a module over the Noetherian regular integral domain $\mathrm{Rees}(\mathrm{I}^\bullet\mathrm{A})$.

Call the subcategory of reflexive sheaves $\mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}})$.

We will aim to show this is equivalent to $\mathsf{Loc}^{\mathrm{cris}}_{\mathbf{Z}_p}(\mathrm{Gal}_K)$ via the étale realization.

For the purpose of the proof, it will be convenient to have several equivalent notions of reflexive at hand.

Proposition 3.5. The following are equivalent:

- A coherent sheaf $\mathcal{E} \in \mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ is isomorphic to its $\mathcal{O}$-linear double dual via the canonical map.
- $\mathcal{E} \in \mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}})$.
- After pullback to $\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}}$ $\mathcal{E} \in \mathsf{Refl}(\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}})$, defined as $M \in \mathsf{Perf}(\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}})$ such that $M|_{\mathcal{O}_{\mathbf{C}_p}^{\mathbb{A}}}$ is locally free and $M|_{\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Nyg}}} \in \mathsf{Perf}(\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Nyg}}) \simeq \mathsf{Perf}_{\mathrm{gr}}(\mathbf{A}_{\mathrm{inf}}\langle u,t\rangle/(ut-\varphi^{-1}(\xi))$ is (u,t)-regular (i.e. $\mathrm{Kos}(M;u,t)$ is discrete).

Proof. Clearly (1) implies (2). To see (2) implies (1), the map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ always exists in $\mathsf{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$, so we can test if it is an equivalence of coherent sheaves after pullback along an fpqc cover. Item (2) says that after pullback to the cover $\mathrm{Spf}\,\mathrm{Rees}(\mathrm{I}^\bullet\mathrm{A})/\mathbf{G}_m$ this morphism becomes an equivalence.

The equivalence of (2) and (3) is more nontrivial. Observe there is a factorization

$$\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Nyg}} \to \mathcal{R}(\mathrm{I}^\bullet\mathrm{A}) \to \mathcal{O}_K^{\mathrm{Nyg}}.$$

Both maps are faithfully flat.

Stating this in terms of rings, it suffices to show that a module is reflexive over $\mathrm{Rees}(\mathrm{I}^\bullet\mathrm{A})$ (condition (2)) if and only if the pullback M to $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{Nyg}}\mathbf{A}_{\mathrm{inf}})$ has the property that $M[1/u], M[1/t]$ are locally free and $\mathrm{Kos}(M;u,t)$ is discrete (condition (3)).

We will use the following lemma to do this:

LEMMA 3.6. Let $(\mathrm{R},\mathfrak{m})$ be a three dimensional regular local ring and let (x,y) be a regular sequence of length 2. Let M be a finitely generated module over R. Then the following are equivalent:

- M is reflexive.
- M has $M[1/x], M[1/y]$ free and $\mathrm{Kos}(M;x,y)$ is discrete.

Proof. Suppose M has $M[1/x], M[1/y]$ free and $\mathrm{Kos}(M;x,y)$ is discrete. Then it follows that $M|_{\mathrm{Spec}\,\mathrm{R}\setminus \mathrm{V}(x,y)}$ is a vector bundle, and discreteness of the Koszul complex means that M is $*$-extended from this restriction. The hypotheses of Stack Project 0E9I are satisfied, as the depth of M_p for $p \in \mathrm{V}(x,y)$ is ≥ 2. Indeed for a Noetherian local ring, M-Koszul regularity and M-regularity are the same, so this says (x,y) is a regular sequence giving a lower bound on the depth (when x,y are in the maximal ideal of $\mathcal{O}_{\mathrm{X},x}$ we can use these, i.e. when $p \in \mathrm{V}(x,y)$). Thus M is reflexive, as $*$-extensions from open subschemes whose complement has codimension ≥ 2 are reflexive.

Conversely, suppose M is reflexive. Then $M[1/x]$ and $M[1/y]$ are reflexive on Noetherian regular schemes of dimension ≤ 2, hence by Stacks Project 0B3N they are locally free. $\square$

The hypotheses apply to $\mathcal{R}(\mathrm{I}^\bullet \mathrm{A})$. Thus we learn that if (2) holds, then (3) must hold after pullback.

Conversely, suppose (3) holds and let M denote the module over $\mathrm{Rees}(\mathrm{I}^\bullet\mathrm{A})$ obtained via pullback. Then by faithfully flat descent for locally free sheaves, $M[1/u]$ and $M[1/t]$ are locally free (the maps $\mathcal{O}^{\mathbb{\Delta}}_{\mathbf{C}_p} \to \mathrm{Spf}\ \mathrm{A} \to \mathcal{O}^{\mathbb{\Delta}}_K$ are also faithfully flat).

As for $\mathrm{Kos}(M; u, t)$ we know its pullback along a faithfully flat map is discrete, so the second condition of the lemma is also satisfied. Indeed, having tor-amplitude in $[a, b]$ is fpqc local. □

We now will make an additional study of $\mathsf{Refl}(\mathcal{O}^{\mathrm{Syn}}_{\mathbf{C}_p})$ – we will need this to show

$$(-)|_{\mathrm{X}^{\mathbb{\Delta}}} : \mathsf{Refl}(\mathcal{O}^{\mathrm{Syn}}_K) \to \mathsf{Vect}^{\varphi}(\mathrm{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$$

is an equivalence.

The first key result is the following.

Proposition 3.7. There is an equivalence of categories

$$(-)|_{\mathrm{X}^{\mathbb{\Delta}}} : \mathsf{Refl}(\mathcal{O}^{\mathrm{Syn}}_{\mathbf{C}_p}) \simeq \mathsf{Vect}^{\varphi}(\mathbf{A}_{\mathrm{inf}}).$$

The latter category are usually called Breuil-Kisin-Fargues modules.

Proof. Let $(\mathbf{A}_{\mathrm{inf}}, \xi)$ be the perfect prism for $\mathcal{O}_{\mathbf{C}_p}$. It suffices to prove a version of this for the Nygaard stack and then identify the Hodge-Tate and de Rham pullbacks. In particular, we will want to show that the category $\mathsf{Isog}(\mathbf{A}_{\mathrm{inf}}, \xi)$ of triples

$$(M, N, \tau)$$

consisting of $M, N \in \mathsf{Vect}(\mathbf{A}_{\mathrm{inf}})$ and an isomorphism $\tau : M[1/\xi] \simeq N[1/\xi]$ is equivalent to the category

$$\mathrm{Refl}_{\mathrm{gr}}(\mathrm{A}\langle u, t\rangle/(ut - \xi))$$

of modules E so that $\mathrm{E}[1/u], \mathrm{E}[1/t]$ are locally free over $\mathbf{A}_{\mathrm{inf}}[u, u^{-1}]$ and $\mathbf{A}_{\mathrm{inf}}[t, t^{-1}]$ respectively and E is (u, t)-regular.

The functor

$$(-)|_{\mathcal{O}^{\mathbb{\Delta}}_{\mathbf{C}_p}} : \mathrm{Refl}_{\mathrm{gr}}(\mathrm{A}\langle u, t\rangle/(ut - \xi) \to \mathsf{Isog}(\mathbf{A}_{\mathrm{inf}}, \xi)$$

sends E to the triple

$$(M(\mathrm{E}), N(\mathrm{E}), \tau)$$

where $M(\mathrm{E}) := \mathrm{E}[1/u]_{\deg 0}$ and $N(\mathrm{E}) := \mathrm{E}[1/t]_{\deg 0}$, equipped with the natural correspondence

$$M(\mathrm{E}) \xleftarrow{u^\infty} \mathrm{E}_{\deg 0} \xrightarrow{t^\infty} N(\mathrm{E})$$

which induces the isogeny τ by perfectness of E.

Now we show this functor has an inverse F, given by equipping objects in $\mathsf{Isog}(\mathbf{A}_{\inf}, \xi)$ with a saturated Nygaardian filtration. Given (M, N, τ), set

$$\mathrm{Fil}^\bullet N := \xi^\bullet M \cap N$$

where the intersection uses the identification $\tau : M[1/\xi] \simeq N[1/\xi]$ to view M as embedded in $N[1/\xi]$. Then we set

$$F(M, N, \tau) := \mathrm{Rees}(\mathrm{Fil}^\bullet N).$$

At the moment, it is unclear why this has the desired properties.

For local freeness of $\mathrm{Rees}(\mathrm{Fil}^\bullet N)[1/u]$ and $\mathrm{Rees}(\mathrm{Fil}^\bullet N)[1/t]$, we just need to use that $\mathrm{Fil}^i N = \xi^i M$ for $i \gg 0$ and $\mathrm{Fil}^i N = N$ for $i \ll 0$. Applying the Rees dictionary implies these identify with $M[u, u^{-1}]$ and $N[t, t^{-1}]$ as desired.

For Koszul regularity, since $\mathrm{Fil}^\bullet N$ is an honest filtration so t acts injectively. To get (t, u) regularity, we just need to check that u acts injectively on $\mathrm{Rees}(\mathrm{Fil}^\bullet N)/t$. This follows from the fact that if $x \in \mathrm{Fil}^i N$ with $\xi \cdot x \in \mathrm{Fil}^{i+1} N$ then $x \in \mathrm{Fil}^{i+1} N$. Using $ut = \varphi^{-1}(\xi)$, it follows that the action of u is injective on $\mathrm{Rees}(\mathrm{Fil}^\bullet N)/t$ and thus $F(M, N, \tau)$ is Koszul regular for (u, t). We omit verification that this module is perfect. This is where the valuation ring property of $\mathcal{O}_{\mathbf{C}_p}$ is used to verify that the resulting object is finitely presented.

To see that F and $(-)|_{\mathcal{O}_{\mathbf{C}_p}^{\mathbb{A}}}$ are mutually inverse equivalences, we show compositions in both directions give the identity. It's obvious that given (M, N, τ) that $\mathrm{Rees}(\mathrm{Fil}^\bullet N)|_{\mathcal{O}_{\mathbf{C}_p}^{\mathbb{A}}} = (M, N, \tau)$. The non-obvious direction is that $F \circ (-)|_{\mathcal{O}_{\mathbf{C}_p}^{\mathbb{A}}}$ is the identity. Let $\mathrm{E} \in \mathsf{Perf}_{\mathrm{gr}}(\mathrm{A}\langle u, t\rangle/(ut - \varphi^{-1}(\xi)))$. By perfectness it is derived (u, t)-complete automatically, so (u, t)-regularity implies E is discrete and u, t act injectively on E.

Now let us consider the triple (M, N, τ) associated to E. We want to show that

$$\mathrm{Rees}(\xi^\bullet M \cap N) = \mathrm{Rees}(\mathrm{E}_{\deg = -\bullet}),$$

as the latter recovers E.

The fact that E is t-regular implies that $\mathrm{E}_{\deg=-\bullet}$ is an honest filtration on the locally free $\mathbf{A}_{\inf}$-module N. Degree shifting, we only need to verify that $\mathrm{E}_{\deg=0} = M \cap N$. Clearly there is an inclusion

$$\mathrm{E}_{\deg=0} \hookrightarrow M \cap N.$$

Conversely, any $x \in M \cap N$ gives a section of $\mathrm{E}_{\mathrm{Spec}\,\mathrm{A}\langle u,t\rangle/(ut-\xi) - \mathrm{V}(u,t)}$, which by regularity is the same as a global section. This is again Stacks Project 0E9I. $\square$

To show the main result, we will to construct an inverse to $(-)|_{\mathcal{O}_K^{\mathbb{A}}}$. In the perfectoid case we see how to do this; the idea will be to study base changes to qrsp rings to specify an inverse using quasisyntomic descent.

Definition 3.8. Let R be a qrsp ring and let M be a prismatic F-crystal in finitely presented Δ_{R}-modules. A Nygaardian filtration is a filtration $\mathrm{Fil}^\bullet M$ in (p, I)-complete modules such that the canonical map

$$M \to \varphi^* M \subset \varphi^* M[1/\mathrm{I}] \simeq M[1/\mathrm{I}]$$

carries $\mathrm{Fil}^i M$ into $\mathrm{I}^i M$ for all $i \in \mathbf{Z}$.

We say a Nygaardian filtration is *saturated* if it is the maximal such filtration (it is the preimage of $\mathrm{I}^i M$ under this map).

Clearly there is a unique saturated Nygaardian filtration on a prismatic F-crystal.

Lemma 3.9. Let $\mathrm{E} \in \mathsf{Perf}(\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}})$, and let R be a p-torsionfree qrsp $\mathcal{O}_{\mathbf{C}_p}$-algebra. For the base change $\mathrm{E}_{\mathrm{R}} \in \mathsf{Perf}(\mathrm{R}^{\mathrm{Syn}})$, the filtration $\mathrm{Fil}^\bullet \mathrm{E}$ is the saturated Nygaardian filtration.

Sketch. To E, we can associate $(M(\mathrm{E}), \mathrm{Fil}^\bullet M(\mathrm{E}), \tilde{\varphi})$. To E_{R}, we associate the (p, I)-completed base change

$$(M(\mathrm{E}_{\mathrm{R}}), \mathrm{Fil}^\bullet M(\mathrm{E}_{\mathrm{R}}), \tilde{\varphi}).$$

To prove the claim, we need to show $\mathrm{Fil}^\bullet$ is an honest filtration (i.e. the maps $\mathrm{Fil}^{i+1} \to \mathrm{Fil}^i$ are injective), and that $\mathrm{Fil}^i M(\mathrm{E}_{\mathrm{R}})$ is obtained as the $\tilde{\varphi}$ preimage of the I-adic filtration.

Fact. $N = \mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E})$ has bounded p-torsion, and so does $N[1/u]/N$.

Since $\mathrm{Fil}^\bullet M(\mathrm{E})$ is saturated we have $\mathrm{Fil}^{-i} M(\mathrm{E}) = M(\mathrm{E})$ for $i \gg 0$ and the same will hold after base change. For the honest filtration claim, we'll just need to check $\mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E}_{\mathrm{R}})$ is coconnective: if $\mathrm{Fil}^{i+1} \to \mathrm{Fil}^i$ had some kernel, since the gr^i are defined as the cones of these morphisms we'll get a non-coconnective cone.

Then

$$\mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E}_{\mathrm{R}}) = \mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E}) \widehat{\otimes}^{\mathrm{L}}_{\mathrm{gr}^\bullet_{\mathrm{Nyg}} \mathbf{A}_{\mathrm{inf}}} \mathrm{gr}^\bullet_{\mathrm{Nyg}} \Delta_{\mathrm{R}}$$

Since $\mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E})$ is discrete and has bounded p-power torsion and we are making a p-completely flat base change, we get coconnectivity.

For the second item, we saw

$$\mathrm{Fil}^{-i} M(\mathrm{E}_{\mathrm{R}}) = M(\mathrm{E}_{\mathrm{R}})$$

for $i \gg 0$. So we just want to know

$$\tilde{\varphi} : \mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E}_{\mathrm{R}}) \to \mathrm{gr}^\bullet_{\mathrm{I}\mathbf{Z}} M(\mathrm{E}_{\mathrm{R}})$$

is injective. The F-gauge structure means

$$\mathrm{gr}^\bullet_{\mathrm{I}\mathbf{Z}} M(\mathrm{E}_{\mathrm{R}}) \simeq \mathrm{gr}^\bullet_{\mathrm{Fil}} M(\mathrm{E}_{\mathrm{R}})[1/u],$$

so putting $N_{\mathrm{R}} = \mathrm{gr}^{\bullet}_{\mathrm{Fil}} M(\mathrm{E})$ we see we want to know $N_{\mathrm{R}}[1/u]/N_{\mathrm{R}}$ is coconnective. But this is a p-completed base change of $N[1/u]/N$ along a p-completely flat map, and this is discrete with bounded p-power torsion. □

With this in place, we can prove the main result.

THEOREM 3.10. There is an equivalence of categories

$$(-)|_{\mathcal{O}_K^{\Delta}} : \mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \to \mathsf{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}).$$

Proof. We show this by constructing an inverse functor. Take the quasisyntomic cover

$$\mathcal{O}_{\mathbf{C}_p} \to \mathcal{O}_K$$

and write $\mathrm{R}^{\bullet}$ for the p-complete Čech nerve.

Now let $\mathcal{C}^i \subset \mathsf{Perf}((\mathrm{R}^i)^{\mathrm{Syn}})$ be the subcategory of prismatic F-gauges with the following conditions:

- The underlying prismatic crystal is a vector bundle.
- The filtration after restricting to $(\mathrm{R}^i)^{\mathrm{Nyg}}$ is a saturated Nygaardian filtration on the underlying prismatic F-crystal.
- The previous two items hold true after base change along any map $\mathrm{R}^i \to \mathrm{R}^j$ in the cosimplicial ring $\mathrm{R}^{\bullet}$.

There is then a natural functor

$$\lim_{\Delta} \mathcal{C}^i \to \lim_{\Delta} \mathsf{Perf}((\mathrm{R}^i)^{\mathrm{Syn}}) \simeq \mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}}).$$

This is termwise fully faithful by construction, so it is fully faithful.

We claim that $\lim \mathcal{C}^i \simeq \mathsf{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$. Indeed, restricting to the associated prismatic F-crystal gives a functor

$$\lim_{\Delta} \mathcal{C}^i \to \lim_{\Delta} \mathsf{Vect}^{\varphi}(\Delta_{\mathrm{R}^i}) \simeq \mathsf{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}).$$

Each functor in the limit is fully faithful, as the saturated Nygaardian filtration is determined by the underlying prismatic F-crystal and preserved by maps between them. Moreover we have already seen that for the $\mathrm{R}^0 = \mathcal{O}_{\mathbf{C}_p}$ we get an equivalence, so the limiting functor is an equivalence. Here we are using that on $\mathrm{R}^0 = \mathbf{A}_{\mathrm{inf}}$, F-gauges whose underlying prismatic crystal is a vector bundle equipped with a saturated Nygaardian filtration are equivalent to $\mathsf{Refl}(\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}})$. Indeed, the inverse functor

$$\mathsf{Vect}^{\varphi}(\mathbf{A}_{\mathrm{inf}}) \to \mathsf{Refl}(\mathcal{O}_{\mathbf{C}_p})^{\mathrm{Syn}}$$

is literally equipping a prismatic F-crystal with a saturated Nygaardian filtration.

This then defines a fully faithful functor

$$\mathsf{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \to \mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \subset \mathsf{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$$

which is quasi-inverse to restriction to $\mathcal{O}_K^{\mathbb{\Delta}}$ by construction. □

COROLLARY 3.11. We have $\mathsf{Refl}(\mathcal{O}_K^{\mathrm{Syn}}) \simeq \mathsf{Loc}_{\mathbf{Z}_p}^{\mathrm{cris}}(\mathrm{Gal}_K)$ via the étale realization.

Proof. This follows by the theorem of Bhatt-Scholze that $\mathsf{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \simeq \mathsf{Loc}_{\mathbf{Z}_p}^{\mathrm{cris}}(\mathrm{Gal}_K)$ via the étale realization functor. Both notions of étale realization are clearly compatible by construction. □

Another corollary you can derive from this is the following:

COROLLARY 3.12. Let $\mathcal{E} \in \mathsf{Perf}(\mathcal{O}_K)^{\mathrm{Syn}})$. After inverting p, every cohomology sheaf of $\mathrm{T}_{\text{ét}}(\mathcal{E})[1/p]$ is crystalline.

This requires some understanding of the kernel of the étale realization, so the proof is omitted.