RIEMANN-HILBERT

1. KATZ'S CORRESPONDENCE

Over $\mathbf{C}$, the first version of the Riemann-Hilbert correspondence is as follows.

> THEOREM 1.1. Let $X/\mathbf{C}$ be a smooth variety. There is an equivalence of categories
> $$\mathsf{Vect}^{\nabla}(X) \simeq \mathsf{Loc}_{\mathbf{C}}(X)$$
> where the left hand side is vector bundles with flat connection and regular singularities and the right hand side is local systems in finite dimensional $\mathbf{C}$-vector spaces. The equivalence is given by
> $$\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_X$$
> and in the other direction $E \mapsto E^{\nabla=0}$, the sheaf of flat sections.

Morally, the key piece in the proof of this theorem will be the exact sequence

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X \longrightarrow 0.$$

For example, by finite étale descent of both sides we may reduce full faithfulness to exactness of this sequence. Moreover, essential surjectivity amounts to showing that there's a set of n flat sections of the rank n vector bundle E so that it is globally generated by these sections. If we know this, then essential surjectivity is obvious. Indeed, flat sections of $\mathcal{L} \otimes \mathcal{O}_X$ recover $\mathcal{L}$, so we just to show every vector bundle with flat connection is of the form $\mathcal{L} \otimes \mathcal{O}_X$. This is exactly the condition about being globally generated by the n flat sections. Flatness is used to deduce $\ker \nabla$ is a local system via Frobenius integrability (also telling the rank of the solution set).

In positive characteristic, the naïve approach will fail, as

$$\mathbf{F}_p \longrightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X$$

will fail to be exact (e.g. the derivative of t^p vanishes). Katz was the first person to give a mod p Riemann-Hilbert correspondence. If one phrases the proof as we just did, the first step is to look for an analogous exact sequence – now we need to get $\mathbf{Z}_p$ or $\mathbf{F}_p$ instead of $\mathbf{C}$. The easiest way to do this is to use the exact sequence

$$0 \longrightarrow \mathbf{Z}_p \longrightarrow W(R) \xrightarrow{\varphi - \mathrm{id}} W(R) \longrightarrow 0.$$

You might then guess the following statement, which is basically a categorified version of this exact sequence.

THEOREM 1.2. Let R be an $\mathbf{F}_p$-algebra. Then

$$\mathrm{D}^{(b)}_{\text{lisse}}(\operatorname{Spec}\mathrm{R}, \mathbf{Z}_p) \simeq \mathsf{Perf}(\mathrm{W}(\mathrm{R}))^{\varphi=1}.$$

The functor in one direction sends $\mathcal{L} \mapsto \mathcal{L} \otimes \mathrm{W}$ (here viewed as an étale sheaf on $(\operatorname{Spec}\mathrm{R})_{\text{ét}}$), and in the other sends $M \mapsto \{M_{\mathrm{R}'}^{\varphi=1}\}$ where the target is the étale sheaf assigning for $\operatorname{Spec}\mathrm{R}'$ étale over $\operatorname{Spec}\mathrm{R}'$ the (derived) Frobenius fixed points of the base change to R'.

This is also t-exact.

Sketch. Full faithfulness of $\mathcal{L} \mapsto \mathcal{L} \otimes \mathrm{W}$ is actually quite easy. By virtue of the functor commuting with shifts and fibers, we can reduce to checking RHom's match where both objects are in the heart. This is essentially the standard argument where if we have a stable category equipped with a non-degenerate t-structure, we can use descending induction to build an object out of shifts and fibers of objects in the heart by starting in the top cohomology. By tensor-Hom adjunction we can make the first object $\mathbf{Z}_p$.

By finite étale descent for both sides of the correspondence, one can reduce this to checking $\mathrm{RHom}(\mathbf{Z}_p, \mathbf{Z}/p^n)$. This then follows from the Artin-Schreier sequence

$$0 \longrightarrow \mathbf{Z}/p^n \longrightarrow \mathrm{W}_n(\mathrm{R}) \xrightarrow{\varphi-\mathrm{id}} \mathrm{W}_n(\mathrm{R}) \longrightarrow 0$$

by using it to compute the RHom on both sides of the correspondence (we may use it to substitute $\operatorname{fib}(\varphi - 1 : \mathrm{W}_n \to \mathrm{W}_n)$ for $\mathbf{Z}/p^n$, and then pull out the limit in the second variable).

Essential surjectivity is a bit more nontrivial. By descending induction from the top cohomology, one may argue each H^i is a vector bundle. The gist is that on the top cohomology, we can take the first nonzero Fitting ideal J and by base change it has $\mathrm{J} = \varphi(\mathrm{J})\mathrm{R}$. One may localize to assume R is a local ring (here we use a technical result: by boundedness both sides are hypersheaves valued in Pr^{L}, so we can check equivalences stalkwise) and use Noetherian approximation to make it Noetherian; then Krull's intersection theorem rules out all possibilities but 0 and R. Indeed, any $x \in \mathrm{J}$ then has $x \in \mathrm{J}^{p^n}$ for all n, and must be 0 if J is not the entire ring.

Once we have deduced all H^i are built out of φ-modules which are vector bundles for some $\mathrm{W}_n(\mathrm{R})$, to deduce essential surjectivity we can study the situation in the heart and use shifts and fibers (noting we already have full faithfulness).

In the vector bundle case, you can use Katz's original proof. Unfortunately this part of the proof is a bit complicated, so we will only say the gist of the idea. Essentially, given an étale φ-module in vector bundles M, since we've reduced to the local ring case, we can assume it's actually free. One may show that after a finite étale base change there exists a Frobenius-invariant basis. This shows we hit every φ-module. ☐

> REMARK 1.3. An understandable confusion is to think that $(\mathrm{R}, \varphi_{\mathrm{R}})$ is not an object of the RHS unless R is perfect (and to increase this confusion, many authors restrict to the perfect case). The reason is that we have to be aware of which side the module structure is coming from on $\varphi^*\mathrm{R}$: left or right matters. For example, if we have any ring map $\varphi : \mathrm{B} \to \mathrm{A}$ inducing
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> $$\varphi : \operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{B}$$
>
> then $\varphi^*\mathcal{O}_{\mathrm{B}} \simeq \mathcal{O}_{\mathrm{A}}$. As modules, $\mathrm{B} \otimes_{\mathrm{B},\varphi} \mathrm{A} \simeq \mathrm{A}$ where we view it as an A-module on the right. Now put $\mathrm{B} = \mathrm{A} = \mathrm{R}$ and φ the Frobenius and we understand better what is happening (and that it has nothing to do with Frobenius).

This is already a great theorem, but we also want to understand constructible sheaves. Over $\mathbf{C}$, the generalization is that

$$\mathrm{Perv}(\mathrm{X}, \mathbf{C}) \simeq \mathrm{DMod}^{\mathrm{hol}}(\mathrm{X})$$

where the LHS are perverse sheaves and the RHS consists of regular holonomic D-modules concentrated in degree zero. This arises from an equivalence of derived categories, and after passing the t-structure on the D-module side through the equivalence one obtains the perverse t-structure on constructible sheaves. This is not the focus on the talk so I won't give further definitions; it's just for orienting yourself in what follows.

The constructible generalization ends up being a great deal harder. In this direction, there are several results but I will focus on two main ones:

- There is a theorem of Bhatt-Lurie which actually goes beyond constructibility, giving a fully faithful functor

  $$\mathsf{Shv}((\operatorname{Spec} \mathrm{R})_{\text{ét}}, \mathbf{F}_p) \to \mathsf{Mod}_{\mathrm{R}}^{\varphi}$$

  and characterize the essential image as *algebraic* φ-modules, where every element is killed by a finite degree polynomial in $\mathrm{R}[\varphi]$. This extends to the derived setting and can be made t-exact. This also works for any ring R. While this is amazing, the problem is that the functor is not very explicit: it's constructed as a left adjoint to an explicit solution functor. Akhil Mathew recently gave a formula, but it involves passing to the perfect site which is again somewhat inexplicit. If one wants to extend these methods to mixed characteristic it's even worse: we need perfectoidization!
- There is also the RH correspondence of Emerton-Kisin. This gives an equivalence of abelian categories

  $$\mathrm{RH} : \mathsf{Perv}(\mathrm{X}_{\text{ét}}, \mathbf{Z}_p) \simeq \mathsf{Mod}_{\mathrm{lfgu}}(\mathrm{D}_{F,\mathfrak{X}})$$

  where lfgu stands for locally finitely generated unit. Here, we insist X is regular and has a formal lift $\mathfrak{X}$. Really it's proved for the derived category, and then is shown to be t-exact for the perverse t-structure. The category of $\mathrm{D}_{F,\mathrm{X}}$-modules is roughly D-modules equipped with Frobenius structure.

2. EMERTON-KISIN

In Emerton-Kisin's original paper, they do not actually deal with $\mathbf{Z}_p$ coefficients. However, if one works through the arguments it is relatively simple to make this extension. In fact both Bhatt-Lurie and Emerton-Kisin give arguments that work in the $\mathbf{Z}/p^n\mathbf{Z}$ case, but never pass to the limit!

Let us first make all the definitions involved clear. Fix the finite field we work over as k. When we work with $\mathbf{Z}_p$ coefficients, we will assume that X/k admits a smooth formal lift $\mathfrak{X}/\operatorname{Spf}\mathrm{W}(k)$ (which need not be the case; for example you can already obstruct lifting to $\mathrm{W}_2(k)$ via Calabi-Yau threefolds).

DEFINITION 2.1. Constructible $\mathbf{Z}_p$ sheaves on X/k are as you'd expect. Gabber defines a perverse t-structure as you'd usually expect: define the middle perversity function as $p(x) = -\dim\overline{\{x\}}$. We set the ≤ 0 part of the perverse t-structure to be constructible $\mathscr{F}$ such that

$$\mathrm{H}^i(i_x^{-1}\mathscr{F}) = 0, i > p(x)$$

and the ≥ 0 part as

$$\mathrm{H}^i(i_x^{!}\mathscr{F}) = 0, i < p(x).$$

The heart of this t-structure is $\mathsf{Perv}(\mathrm{X}_{\text{ét}}, \mathbf{Z}_p)$. These define a t-structure, as shown by Gabber.

Now we turn to the other side. Assume X/k is affine for now. If $\mathfrak{X}/\operatorname{Spf}\mathrm{W}(k)$ is a formal lift of X with a Frobenius lift φ, we set for $\mathrm{X} = \mathbf{A}_k^n$

$$\mathrm{D}_{\mathfrak{X}} = \mathrm{W}(k)\langle \mathrm{T}_1, \ldots, \mathrm{T}_n, \partial_{\mathrm{T}_1}^{[m_1]}, \ldots, \partial_{\mathrm{T}_n}^{[m_n]}\rangle$$

to be the algebra of infinitesimal differential operators (so we give the derivations in the Weyl algebra divided powers). This globalizes to the sheaf of infinite level differential operators. We let $\mathrm{D}_{F,\mathrm{X}}$ denote the sheaf of rings given by adjoining an indeterminate F (for the Frobenius) with some relations:

- $F\mathrm{T}_i = \varphi(\mathrm{T}_i)F$
- $\partial_{\mathrm{T}_i}F = \partial_{\mathrm{T}_i}(\varphi(\mathrm{T}_i))F\partial_{\mathrm{T}_i}$.

Due to the ∂_{T_i} admitting divided powers, there is a change of Frobenius formula that shows $\mathrm{D}_{F,\mathrm{X}}$ does not depend on φ. Therefore it globalizes to any X admitting a formal lift $\mathfrak{X}$, without needing a Frobenius lift.

Now we let $\mathsf{Mod}_{\mathrm{lfgu}}(\mathrm{D}_{F,\mathfrak{X}})$ denote the locally finitely generated unit modules. The first part is self-explanatory: locally, the module is finitely generated over $\mathrm{D}_{F,\mathfrak{X}}$ viewed as a sheaf of noncommutative rings. The second part asks that there is an isomorphism

$$F^*M \simeq M,$$

or that it is unit. This is essentially the condition we called étale before; here we make sense of F^*M by making the tensor product over the module map $F : \mathrm{D}_{F,\mathfrak{X}} \to \mathrm{D}_{F,\mathfrak{X}}$ (which globalizes).

By virtue of working with infinitesimal crystals, with a bit of additional work one may remove the requirement of a lift $\mathfrak{X}$ (we only need to prove the local version of the correspondence where a lift always exists).

Remark 2.2. The infinitesimal differential operators might be surprising, as usually we want crystalline differential operators in positive characteristic. However, the motivation is actually quite simple: before involving the Frobenius, infinitesimal cohomology of $\mathrm{R}\Gamma_{\mathrm{inf}}(\mathrm{X}/\mathrm{W})$ for smooth proper X over $\bar{k}$ identifies with $\mathrm{R}\Gamma_{\text{ét}}(\mathrm{X}, \mathbf{Z}_p)$. This is the primary reason that infinitesimal crystals are more convenient.

Note however that one could also work with the crystalline site with no issues. For X regular over k we have an equivalence

$$\mathrm{D}_\omega(\mathrm{X}_{\mathrm{inf}\,/\mathrm{W}}, \mathcal{O}_{\mathrm{X}/\mathrm{W}})^{\varphi=1} \simeq \mathrm{D}_\omega(\mathrm{X}_{\mathrm{cris}/\mathrm{W}}, \mathcal{O}_{\mathrm{X}/\mathrm{W}})^{\varphi=1}$$

on subcategories of compact objects in $\mathrm{D}(\mathrm{X}_{\mathrm{inf}}/\mathrm{W}, \mathcal{O})^{\varphi=1}$ and $\mathrm{D}(\mathrm{X}_{\mathrm{cris}}/\mathrm{W}, \mathcal{O})^{\varphi=1}$. Moreover this is compatible with cohomology: $\mathrm{R}\Gamma_{\mathrm{inf}}(\mathrm{X}/\mathrm{W}) = \lim_\varphi \mathrm{R}\Gamma_{\mathrm{cris}}(\mathrm{X}/\mathrm{W})$. In the smooth proper case, $\bigcap_n \varphi^n \mathrm{H}^i_{\mathrm{cris}}(\mathrm{X}/\mathrm{W})$ gives $\mathrm{H}^i_{\mathrm{inf}}$, or the unit root part. Thus when one further takes Frobenius invariants, we get the same result.

When one works modulo p with $\mathbf{F}_p$-coefficients, the situation is easier. We will use this later when deducing full faithfulness. Note that in what follows $\mathcal{O}_{F,\mathrm{X}}$ itself is not unit (which is why the proposition can make sense).

Proposition 2.3. There is an equivalence of categories $\mathrm{D}_{\mathrm{unit}}(\mathrm{D}_{F,\mathrm{X}}) \simeq \mathrm{D}_{\mathrm{unit}}(\mathcal{O}_{F,\mathrm{X}})$ when X is a smooth k-scheme.

Proof. Work locally, and assume X is affine, and let M be a unit $\mathcal{O}_{F,\mathrm{X}}$-module. The claim is essentially that

$$\mathrm{D}_{F,\mathrm{X}} \otimes^{\mathrm{L}}_{\mathcal{O}_{F,\mathrm{X}}} M \simeq M.$$

By taking a resolution in $\mathcal{O}_{F,\mathrm{X}}$-modules, we can reduce to M being free but with possibly non-unit Frobenius structure. One may fix this with a two term resolution

$$\mathcal{O}^{\mathrm{I}}_{F,\mathrm{X}} \xrightarrow{1-\mu F} \mathcal{O}^{\mathrm{I}}_{F,\mathrm{X}} \longrightarrow M.$$

Now the claim amounts to checking the complex $1-\mu F : \mathcal{O}^{\mathrm{I}}_{F,\mathrm{X}} \to \mathcal{O}^{\mathrm{I}}_{F,\mathrm{X}}$ is quasi-isomorphic to the analogous complex $1-\mu F : \mathrm{D}^{\mathrm{I}}_{F,\mathrm{X}} \to \mathrm{D}^{\mathrm{I}}_{F,\mathrm{X}}$ via extension of scalars. It is easy to see both maps are injective. For matching the cokernels, we must show a differential operator in $\mathrm{D}_{F,\mathrm{X}}$ can be written as

$$a + b(1-\mu F)$$

where $a \in \mathcal{O}_{F,\mathrm{X}}$ and $b \in \mathrm{D}_{F,\mathrm{X}}$. This can be shown by descending induction on the order of differential operators, and lets us deduce the map of cokernels is surjective. To be injective, we need

$$\mathcal{O}_{F,\mathrm{X}} \cap \mathrm{D}_{F,\mathrm{X}}(1-\mu F) = \mathcal{O}_{F,\mathrm{X}}(1-\mu F).$$

This may also be argued by descending induction on the order of differential operators. □

In particular, for $\mathbf{F}_p$ coefficients one may completely ignore the D-module aspect so it is much better to work with $\mathcal{O}_{F,\mathrm{X}}$-modules to have simpler definitions.

This defines both sides of the equivalence. Next, we construct the functor.

> DEFINITION 2.4. Let X be a Noetherian regular k-scheme with lift $\mathfrak{X}/\operatorname{Spf} \mathrm{W}(k)$. There is a functor
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> $$\mathrm{RH} : \mathrm{D}^b_{\mathrm{ctf}}(\mathfrak{X}_{\text{ét}}, \mathbf{Z}_p)^{\mathrm{op}} \to \mathrm{D}^b(\mathrm{D}_{F,\mathfrak{X}})$$
>
> given by $\mathrm{RHom}_{\mathrm{D}(\mathfrak{X}_{\text{ét}}, \mathbf{Z}_p)}(-, \mathbf{G}_a)$. We endow this with a $\mathrm{D}_{F,\mathfrak{X}}$-module structure induced by the Frobenius and connection on $\mathbf{G}_a$.
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> There is also a solution functor
>
> $$\mathrm{Sol} : \mathrm{D}^b_{\mathrm{lfgu}}(\mathrm{D}_{F,\mathfrak{X}})^{\mathrm{op}} \to \mathrm{D}(\mathfrak{X}_{\text{ét}}, \mathbf{Z}_p)$$
>
> defined by $\mathrm{RHom}_{\mathrm{D}_{F,\mathfrak{X}_{\text{ét}}}}(-, \mathbf{G}_a)$.

It turns out RH actually outputs finitely generated unit modules, but this isn't actually obvious.

> THEOREM 2.5. Let X be a Noetherian regular k-scheme with formal lift $\mathfrak{X}$. Then RH and Sol induce an antiequivalences
>
> $$\mathrm{D}^b_{\mathrm{cons}}(\mathrm{X}, \mathbf{Z}_p)^{\mathrm{op}} \simeq \mathrm{D}^b_{\mathrm{lfgu}}(\mathrm{D}_{F,\mathfrak{X}}).$$

> Moreover, upon shifting the solution functor by $d = \dim \mathrm{X}$ the natural t-structure on the right hand side corresponds to Gabber's perverse t-structure on the left under the antiequivalence.

We've been very abstract so far, so we are in desperate need of examples.

> Example 2.6. Let $\mathrm{X} = \mathbf{A}^1_{\mathbf{F}_p}$, so $\mathfrak{X} = \operatorname{Spf} \mathbf{Z}_p\langle \mathrm{T}\rangle$. The following are some useful $\mathrm{D}_{F,\mathfrak{X}}$-modules. Note that we have not performed the perversity shift.
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> - $\mathcal{O}_{\mathfrak{X}}$ equipped with its natural Frobenius and action of $\partial_{\mathrm{T}}^{[n]}$. This corresponds to $\mathbf{Z}_p$.
> - $\mathcal{O}_{\mathfrak{X}}\langle 1/\mathrm{T}\rangle$. The operators $\partial_{\mathrm{T}}^{[n]}$ are again the natural divided power nth derivatives on sections, and Frobenius sends $\mathrm{T} \mapsto \mathrm{T}^p$. It is lfgu because we can generate it as a Frobenius module by $\frac{1}{\mathrm{T}^i}$ for $i = 0$ to $p - 1$. It corresponds to $j_! \mathbf{Z}_p$ where $j : \mathbf{G}_m \to \mathbf{A}^1_{\mathbf{F}_p}$ is the inclusion.
> - We also have $\mathcal{O}_{\mathfrak{X}}\langle 1/\mathrm{T}\rangle/\mathcal{O}_{\mathfrak{X}}[-1]$. This corresponds to $i_* \mathbf{Z}_p$ where i is the inclusion of $\{\mathrm{T} = 0\}$ into $\mathbf{A}^1_{\mathbf{F}_p}$. Note that this is extremely natural to associate to this, as $\operatorname{fib}(j_! \mathbf{Z}_p \to \mathbf{Z}_p)$ gives $i_* \mathbf{Z}_p$, and so if we want our functor to be compatible with fibers we assign this.
> - One can also do things like Beilinson gluing for perverse sheaves, and pass this through the correspondence by defining unipotent nearby cycles on the D-module side. The analogous theory over $\mathbf{C}$ also exists.

Now we turn to Akhil Mathew's proof, since the Emerton-Kisin proof is much more involved (it would not fit in a single talk without omitting most of the argument). First, we need some preliminaries. The first important observation Akhil Mathew makes is that it's more convenient to define Sol using the big étale site.

> Definition 2.7 (Solutions revisited). Let $\mathfrak{X} = \operatorname{Spf} \mathrm{R}$ be our smooth formal scheme, and $\mathrm{fSch}_{\mathfrak{X}}$ the big étale site over $\mathfrak{X}$, restricted to smooth formal schemes. Then we can define
>
> $$\mathrm{Sol}^{\mathrm{big}} : \mathrm{D}(\mathrm{D}_{F,\mathfrak{X}})^{\mathrm{op}} \to \mathrm{D}(\mathrm{fSch}_{\mathfrak{X}}, \mathbf{Z}_p)$$
>
> carrying M to the functor sending a p-complete R-algebra R′ to $\mathrm{RHom}_{\mathrm{D}_{F,\mathfrak{X}}}(M, \mathcal{O}_{\operatorname{Spf} \mathrm{R}'}) \in \mathrm{D}^\wedge(\mathbf{Z}_p)$. This defines a hypercomplete étale sheaf valued in $\mathrm{D}^\wedge(\mathbf{Z}_p)$ on $\mathrm{fSch}_{\mathfrak{X}}$. Pushing forward to the small étale site, it is compatible with the previous definition. It will turn out that
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> $$\lambda_* \mathrm{Sol}^{\mathrm{big}} = \mathrm{Sol}$$
>
> when we restrict to lfgu modules.

> We will also use this for $\mathcal{O}_{F,\mathrm{X}}$-modules; there we can make the same definition, but it is harmless to extend to all schemes. Note that we can make the restriction to smooth formal schemes issue go away entirely by just applying the fully faithful pullback.

On objects with bounded projective amplitude, this functor is compatible with base change. This is because it sends $\bigoplus_{\mathrm{I}} \mathrm{D}_{F,\mathfrak{X}}$ to $\prod_{\mathrm{I}} \mathbf{G}_a$, so after resolving we just need to check f^* carries $\prod_{\mathrm{I}} \mathbf{G}_a$ to $\prod_{\mathrm{I}} \mathbf{G}_a$.

In what follows we will frequently use the notion of a thick subcategory generated by an object.

> PROPOSITION 2.8. Assume $\mathfrak{X}$ is smooth. The restriction of $\mathrm{Sol}^{\mathrm{big}}$ to unit objects with bounded projective amplitude is fully faithful.

Proof. Suppose we are considering $\mathrm{RHom}_{\mathrm{D}_{F,\mathfrak{X}}}(N, M)$ and want to match it with

$$\mathrm{RHom}_{\mathrm{D}(\mathrm{fSch}_{\mathfrak{X}},\mathbf{Z}_p)}(\mathrm{Sol}^{\mathrm{big}}(M), \mathrm{Sol}^{\mathrm{big}}(N)).$$

By derived p-completeness we can reduce modulo p to check the equivalence. By virtue of the unit condition and the previous proposition this allows us to use $\mathcal{O}_{F,\mathrm{X}}$-modules and $\mathrm{D}(\mathrm{Sch}_{\mathrm{X}}, \mathbf{F}_p)$. In this case there is also no harm in changing from smooth schemes to all schemes in the big étale site. We can now reduce to checking $N = \mathcal{O}_{F,\mathrm{X}}$, by using the fact that both sides carry colimits to limits.

In this case, we then reduce to the case where $\mathrm{X} = \operatorname{Spec} \mathrm{R}$ is affine so $N = \mathrm{R}[F]$, and then we will want to replace R by $\mathrm{R}_{\mathrm{perf}}$ which is trickier to justify. Kunz's theorem tells us the Frobenius is faithfully flat, so we get a flat cover $\operatorname{Spec} \mathrm{R}_{\mathrm{perf}} \to \operatorname{Spec} \mathrm{R}$. We claim that the map

$$\mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R}},\mathbf{F}_p)}(\mathrm{Sol}^{\mathrm{big}}(M), \mathbf{G}_a) \to \mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R},\mathrm{perf}},\mathbf{F}_p)}(\mathrm{Sol}^{\mathrm{big}}(\mathrm{R}_{\mathrm{perf}} \otimes_{\mathrm{R}} M), \mathbf{G}_a)$$

exhibits the target as the extension of scalars of the source by $\mathrm{R}_{\mathrm{perf}}$ (then allowing us to reduce to the perfect case by descent). We can reduce to checking for $\prod_{\mathrm{I}} \mathbf{G}_a$, as by definition of M (namely bounded projective dimension) we know $\mathrm{Sol}^{\mathrm{big}}(M)$ lies in the thick subcategory generated by these objects. Thus, we reduce to checking

$$\mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R}},\mathbf{F}_p)}(\prod_{\mathrm{I}} \mathbf{G}_a, \mathbf{G}_a) \to \mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R},\mathrm{perf}},\mathbf{F}_p)}(\prod_{\mathrm{I}} \mathbf{G}_a, \mathbf{G}_a)$$

is extension of scalars by $\mathrm{R}_{\mathrm{perf}}$. One can check that the pushforward from the perfect big étale site to the big étale site of $\mathbf{G}_a$ is $\mathbf{G}_a \otimes_{\mathrm{R}} \mathrm{R}_{\mathrm{perf}}$. Since $\prod_{\mathrm{I}} \mathbf{G}_a$ is pseudocoherent and $\mathrm{R}_{\mathrm{perf}}$ is a filtered colimit of finite free R-modules by Lazard's theorem, the desired claim follows. Here, pseudocoherence ensures the derived Hom commutes with filtered colimits.

This allows us to use flat descent to reduce to checking the assertion for $\mathrm{R}_{\mathrm{perf}}$, as on the module side this is obviously what happens.

Now we turn to the perfect case. Since Frobenius is invertible, the $\mathrm{R}_{\mathrm{perf}}[F]$ module $\mathrm{R}_{\mathrm{perf}} \otimes_{\mathrm{R}} M$ comes from a $\mathrm{R}_{\mathrm{perf}}[F^{\pm}]$ module by restriction. This restriction functor is moreover fully faithful, so we may regard $\mathrm{R}_{\mathrm{perf}} \otimes_{\mathrm{R}} M$ as a $\mathrm{R}_{\mathrm{perf}}[F^{\pm}]$ module without loss of generality. The desired claim is then that

$$\mathrm{R}_{\mathrm{perf}} \otimes_{\mathrm{R}} M \to \mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R,perf}},\mathbf{F}_p)}(\mathrm{Sol}^{\mathrm{big}}(\mathrm{R}_{\mathrm{perf}} \otimes_{\mathrm{R}} M), \mathbf{G}_a)$$

is an equivalence. The same argument as before reduces us to checking whether

$$\mathrm{RHom}_{\mathrm{D}(\mathrm{Sch}_{\mathrm{R,perf}},\mathbf{F}_p)}(\prod_{\mathrm{I}} \mathbf{G}_a, \mathbf{G}_a) \simeq \bigoplus_{\mathrm{I}} \mathrm{R}_{\mathrm{perf}}[F^{\pm}].$$

We may then use Breen's calculation of $\mathrm{RHom}(\mathbf{G}_a, \mathbf{G}_a)$ in the big perfect site (as just $\mathrm{R}_{\mathrm{perf}}[F^{\pm}]$). □

It is easy to check $\mathrm{Sol}^{\mathrm{big}}$ has a left inverse given by

$$\mathrm{RH}^{\mathrm{big}} := \mathrm{RHom}_{\mathrm{D}(\mathrm{fSch}_{\mathfrak{X}},\mathbf{Z}_p)}(-, \mathbf{G}_a).$$

Indeed, one may use the case of the result for $N = \mathrm{D}_{F,\mathfrak{X}}$ to deduce this.

> Remark 2.9. The big étale sheaf $\mathbf{G}_a$ is not in the essential image of λ^*. We will crucially use $\mathbf{G}_a$ several times in the argument, which makes it important we use the big étale site.
>
> Note also that the small étale sheaf $\mathbf{G}_a$ is not pseudocoherent in $\mathrm{D}_{\text{ét}}(\mathrm{X}, \mathbf{F}_p)$, and that Breen's RHom computation we use at the end will fail dramatically in the small étale site (it will have nonzero Ext^{2n}).

Now we can see the rest of the proof.

Proof. We start by producing a factorization of $\mathrm{Sol}^{\mathrm{big}}$ on lfgu modules. Namely, we want to argue that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{D}^b_{\mathrm{lfgu}}(\mathrm{D}_{F,\mathfrak{X}}) \arrow[r, "\mathrm{Sol}"] \arrow[rr, bend left, "\mathrm{Sol}^{\mathrm{big}}"] & \mathrm{D}^b_{\mathrm{cons}}(\mathrm{X}, \mathbf{Z}_p) \arrow[r, "\lambda^*"] & \mathrm{D}_{\text{ét}}(\mathrm{fSch}_{\mathfrak{X}}, \mathbf{Z}_p)
\end{tikzcd}$$

For $\mathrm{D}_{F,\mathfrak{X}}$-modules there's no reason for the output of big solutions to come from the small étale site so we should not expect such a factorization. Further, it's also not obvious Sol outputs constructible sheaves.

For the constraint on the essential image of big solutions, we need the following:

- First, argue $\mathrm{Sol}^{\mathrm{big}}(M)$ for M lfgu comes as $\lambda^*\mathscr{F}$ pulled back from the small étale site. The key idea here is that $\mathrm{im}(\lambda^*)$ is objects commuting with filtered colimits and taking local homomorphisms of strictly Henselian local rings to equivalences, so one shows that the output of big solutions has this property. It's easy to check any pullback of an étale sheaf to the big site has these properties; conversely, if a big étale sheaf $\mathscr{G}$ has these properties then

  $$\lambda^*\lambda_*\mathscr{G} \simeq \mathscr{G}.$$

  Colimit compatibility and étale descent allow us to identify stalks on both sides using the strict henselian property.

  To deduce big solutions have this property, roughly one may first reduce to where our lfgu module M discrete and then argue we can reduce to M the unitalization of a finite free $\mathcal{O}_{\mathfrak{X}}$-module. For this specific case, the universal property of unitalization identifies the big solutions as sending

  $$\mathrm{R}' \mapsto [1 - \mu\varphi : \mathrm{R}'^n \to \mathrm{R}'^n].$$

  Here μ is just some matrix encoding a Frobenius semilinear map. Colimit compatibility is clear, and the strict henselian property can be reduced to checking the case of the map to the residue field of a strict henselian local ring. There is an étale self map of $\mathbf{A}^n_{\mathrm{R}}$ where on R' it acts by $1 - M\varphi$, so the étale lifting property deduces we claim.

- Once we have shown this, argue that $\mathscr{F}$ is actually constructible. This follows by checking all truncations are compact in $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathbf{Z}_p)$ using that $\mathrm{Sol}^{\mathrm{big}}$ lies in the thick subcategory generated by $\prod_{\mathrm{I}} \mathbf{G}_a$ and using the characterization of compact objects in $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathbf{Z}_p)$ as constructible sheaves. In particular $\prod_{\mathrm{I}} \mathbf{G}_a$ is pseudocoherent, so the preimage of big solutions is compact in any truncation.

The factorization follows from the first point. We have $\mathrm{Sol} = \lambda_*\mathrm{Sol}^{\mathrm{big}}$, and applying λ^* to both sides we just need the identity

$$\lambda^*\lambda_*\mathrm{Sol}^{\mathrm{big}} = \mathrm{Sol}^{\mathrm{big}}$$

on lfgu modules. This is covered by the first point.

Since λ^* is fully faithful and the composite $\mathrm{Sol}^{\mathrm{big}}$ is as well, it follows that Sol is fully faithful.

We next need to understand the essential image. It would suffice to show that the essential image of $\mathrm{Sol}^{\mathrm{big}}$ on lfgu modules contains $\lambda^*\mathscr{F}$ for any constructible $\mathscr{F}$. That is, we need the following claim.

Claim. For any constructible sheaf $\mathscr{F}$ there exists $M_{\mathscr{F}}$ which is locally finitely generated unit such that $\lambda^*\mathscr{F} \simeq \mathrm{Sol}^{\mathrm{big}}(M_{\mathscr{F}})$.

We now turn to proving this claim. The key fact is that for our X, we can generate $D^b_{\mathrm{cons}}(X, \mathbf{Z}_p)$ as a thick subcategory by $f_*\mathbf{Z}_p$ over finite finitely presented morphisms f. The gist of this idea is that we just need to explain how to produce $j_!\mathcal{L}$, and for these we may reduce to showing we have $h_!\mathbf{Z}_p$ for h the composite of a finite étale cover of $U \subset X$ and the inclusion of the open; Zariski's main theorem says that we can write this as an open map $V \to X'$ followed by a finite map $f : X' \to X$. Let $Z = V^c$, so then

$$h_!\mathbf{Z}_p \simeq \mathrm{fib}(f_*\mathbf{Z}_p \to (f|_Z)_*\mathbf{Z}_p).$$

Now we see that it suffices to look at just finite finitely presented morphisms. Obviously the claim is true for $\mathscr{F} = \mathbf{Z}_p$, so this reduces us to verifying pushforward compatibility in a limited sense which we omit.

At this point, we have now deduced that Sol is an equivalence. However, we are also interested in the functor RH. Recall $\mathrm{RH}^{\mathrm{big}}$ was a left inverse to big solutions, so let us update the diagram.

$$\begin{tikzcd}
D^b_{\mathrm{lfgu}}(D_{F,\mathfrak{X}}) \arrow[r, "\mathrm{Sol}", "\sim"'] \arrow[rr, bend left, "\mathrm{Sol}^{\mathrm{big}}", "\sim"'] & D^b_{\mathrm{cons}}(X, \mathbf{Z}_p) \arrow[r, "\lambda^*", "\sim"'] \arrow[l, bend left, "\mathrm{RH}"] & \lambda^* D^b_{\mathrm{cons}}(X, \mathbf{Z}_p) \arrow[ll, bend left, "\mathrm{RH}^{\mathrm{big}}"]
\end{tikzcd}$$

By abuse of notation we use $\lambda^* D^b_{\mathrm{cons}}(X, \mathbf{Z}_p)$ to denote the essential image of λ^* on constructible sheaves.

This still commutes: one may check

$$\mathrm{RH}^{\mathrm{big}}(\lambda^*\mathscr{F}) \simeq \mathrm{RH}(\mathscr{F})$$

which follows since $\mathbf{G}_a$ is pushed forward from the big étale site (then $\lambda^*\lambda_*\mathbf{G}_a \simeq \mathbf{G}_a$ and we can use full faithfulness of λ^*).

The functor $\mathrm{RH}^{\mathrm{big}}$ is a left inverse to $\mathrm{Sol}^{\mathrm{big}}$, hence an equivalence. We deduce RH is a left inverse to Sol and an equivalence. ☐

> REMARK 2.10. To be perverse t-exact, we have to introduce a global shift in the definition of RH by the dimension d of X over $\mathbf{F}_p$. This way $\mathcal{O}_{\mathfrak{X}}$ is sent to $\mathbf{Z}_p[d]$, which is now perverse.

> Remark 2.11. We made the argument in the smooth case here. The regular case also follows for $\mathbf{F}_p$ coefficients by Popescu's theorem (past this it's a bit tricky as we need to deal with choices of lifts).

We just saw a very abstract version of this, so I think to end I want to give some example computations you can do with this functor. I will focus on what happens for $\mathbf{A}^1$, as this captures most of what we need.