# DERIVED PRISMATIC COHOMOLOGY

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## 1. Failure with singularities

So far, we have seen prismatic cohomology only on smooth $p$-adic formal schemes.

This assumption is essential! For example, in the proof of the Hodge-Tate comparison, even in characteristic $p$ the étale localization step to a polynomial algebra was essential. You cannot ensure that the Frobenius on $\mathrm{A}/p$ is flat to get the untwisting via base change of prisms unless we do the localization to a polynomial algebra first.

So, we see that for the methods of the argument the smoothness was important. This does still not eliminate the possibility that some results could hold outside of the smooth case. However, even with mild singularities we cannot have all comparison theorems hold as the following example demonstrates.

Example 1.1. The issue can already be seen in crystalline cohomology. Consider a proper lci $\mathbf{F}_p$-scheme $\mathfrak{X}$, here specifically a cuspidal cubic.

We'll run a hypothetical calculation assuming that all comparison results hold past the smooth case. We see

$$\mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathrm{W}(\mathbf{F}_p)) \otimes^{\mathbb{L}}_{\mathrm{W}} \mathbf{F}_p$$

is the same as a Frobenius twist of $\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}/\mathbf{F}_p)$. Note that this tensor product is underived, as the complex is $\mathrm{W}$-flat.

This is fine, but we'll run into trouble with the Hodge-Tate comparison now. The Hodge-Tate comparison would have us identify this with the de Rham complex when we take $\mathrm{H}^i$ and twist. But in this particular example, $\mathrm{H}^2$ is actually *infinitely* generated over $\mathbf{F}_p$ by a result of Bhatt. This will not be the case for $\Omega^2$, which you can write down explicitly: we have

$$\Omega^1|_{\mathrm{Spec}\,\mathbf{F}_p[x,y]/(x^2-y^3)} = \frac{\mathbf{F}_p \oplus \mathbf{F}_p}{(2x, 3y)}.$$

Then $\Omega^2$ will not be infinitely generated over $k$.

REMARK 1.2. We also know that $H^1_{\mathrm{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/W(\mathbf{F}_p))$ is infinitely generated. However,

$$H^1_{\mathrm{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/W(\mathbf{F}_p)) \otimes_{W(\mathbf{F}_p)} \mathbf{F}_p$$

will no longer be infinitely generated (e.g. by de Rham-Witt). So smoothness still saves you.

## 2. Kan extension

The usual solution to this issue is to Kan extend everything that we have in the smooth case. The case we will deal with falls under the name of "non-abelian derived functors".

Last semester, Daishi talked about the cotangent complex and how Quillen's formalism of non-abelian left derived functors can allow for a useful extension of $\Omega^i$ past the smooth case. We will want to apply the same procedure to prismatic cohomology.

DEFINITION 2.1 (Sifted colimits). Let $\mathcal{D}$ be a category. Then we call $\mathcal{D}$ *sifted* if $\mathcal{D}$-colimits in Set commute with finite products (for a proper $\infty$-categorical definition, you ask $\mathcal{D} \to \mathcal{D} \times \mathcal{D}$ to be cofinal, i.e. precomposition with it preserves colimits).

A sifted colimit in a category $\mathcal{C}$ is a colimit over a diagram $\mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is sifted.

We write $\mathsf{Fun}_\Sigma(-,-)$ for functors commuting with sifted colimits.

EXAMPLE 2.2. Any filtered category is sifted.

The opposite simplex category $\Delta^{\mathrm{op}}$ is a sifted category which is not filtered. Thus, geometric realizations are sifted colimits.

To see this is sifted in the $\infty$-categorical sense, one shows $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal. For this, one proves the equivalent assertion that

$$\Delta^{\mathrm{op}} \times_{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})_{[m]\times[n]/}$$

is contractible. This is the opposite category of the category $\mathcal{C}$ of objects $[p]$ with maps $\{[p] \to [n], [p] \to [m]\}$, so it suffices to show this is contractible. There's an adjunction (therefore a homotopy equivalence) to the category of monomorphisms $[p] \to [m] \times [n]$ by mapping to $\operatorname{im} f \to [m] \times [n]$ where $f : [p] \times [m] \times [n]$ is the original map in $\mathcal{C}$ (and in the other way, the forgetful functor). It is possible to see the homotopy type of this final category is $\Delta^n \times \Delta^m$, so it is indeed contractible.

We are now ready to understand non-abelian derived functors.

DEFINITION 2.3 (Non-abelian derived functors). Let $F : \mathsf{Poly}_{\mathrm{A}} \to \mathcal{C}$ be a functor landing in an $\infty$-category $\mathcal{C}$ admitting all colimits. Taking left derived functors yields an equivalence

$$\mathsf{Fun}(\mathsf{Poly}_{\mathrm{A}}, \mathcal{C}) \simeq \mathsf{Fun}_{\Sigma}(\mathsf{sAlg}_{\mathrm{A}}, \mathcal{C}).$$

In other words, functors on polynomial algebras admit extensions to $\mathsf{sAlg}_{\mathrm{A}}$ uniquely determined by the requirement that the extension commutes with sifted colimits.

The equivalence is given by sending $F \mapsto \mathrm{L}F$, the left Kan extension of $F : \mathsf{Poly}_{\mathrm{A}} \to \mathcal{C}$ along the inclusion $\mathsf{Poly}_{\mathrm{A}} \subset \mathsf{sAlg}_{\mathrm{A}}$.

Here, we use the $\infty$-category of simplicial commutative rings. In this setting, a simplicial commutative $\mathrm{A}$-algebra is a presheaf $\mathrm{Poly}_{\mathrm{A}}^{\mathrm{op}} \to \mathcal{S}$ sending finite coproducts in Poly to products of spaces.

REMARK 2.4. For nice $\mathcal{C}$ and $\mathcal{D}$, we clearly have

$$\mathsf{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathsf{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathcal{C}), \mathcal{D})$$

where category $\mathcal{P}_{\Sigma}(\mathcal{C})$ is given by *freely* adjoining filtered colimits. Formally, this is defined as the subcategory of presheaves (in spaces) on $\mathcal{C}$ spanned by finite product preserving functors, but even from this perspective the claim is not difficult to prove.

By definition $\mathcal{P}_{\Sigma}(\mathsf{Poly}_{\mathrm{A}})$ yields $\mathsf{sAlg}_{\mathrm{A}}$, so the real content is that the $\infty$-categorical localization of the ordinary category of simplicial commutative rings yields this category.

To compute the total derived functor, one uses a projective resolution just like in the usual case with $\mathrm{D}(\mathrm{R})$. Here, this takes the form

$$\mathrm{L}F(\mathrm{B}) \simeq |\mathrm{L}F(\mathrm{P}^{\bullet})|$$

where $\mathrm{P}^{\bullet} \to \mathrm{B}$ is some simplicial polynomial algebra resolution. Indeed, geometric realization $|\cdot|$ is a sifted colimit, and by the determining requirement of $\mathrm{L}F$ the formula follows. Even with infinitely many generators, $\mathrm{L}F(\mathrm{A}[S]) = \mathrm{colim}_{S' \subset S}\, \mathrm{L}F(\mathrm{A}[S'])$ over finite $S'$ where we already know the value of the functor.

EXAMPLE 2.5 (Abelian derived functors). Let $\mathcal{A}$ be a normal abelian category. An object $\mathrm{P} \in \mathcal{A}$ is projective if and only if $\mathrm{Hom}(\mathrm{P}, -)$ commutes with geometric realization of simplicial objects.

Assume $\mathcal{A}$ has enough projectives. Then

$$\mathcal{P}_\Sigma(\mathcal{A}^{\mathrm{comp,proj}}) \simeq \mathrm{D}_{\geq 0}(\mathcal{A}).$$

The previous formalism taken with a functor $\mathcal{A} \to \mathsf{Ab} \hookrightarrow \mathrm{D}_{\geq 0}(\mathsf{Ab})$ is then *actually* computing the usual derived functor

Example 2.6. Take $\Omega^i_{\mathrm{B/A}}$ as a functor $\mathsf{Poly}_\mathrm{A} \to \mathrm{D(A)}$. Then the cotangent complex $\Lambda^i \mathrm{L}_{\mathrm{B/A}}$ is the nonabelian derived functor. If $i = 1$ we simply mean $\mathrm{L}_{\mathrm{B/A}}$, otherwise $\Lambda^i$ actually means $\mathrm{L}\Lambda^i$, the nonabelian derived functor of exterior powers.

An important fact we will use is that $\mathrm{L}_{\mathrm{B/A}}$ agrees with $\Omega_{\mathrm{B/A}}$ when $\mathrm{B}$ is smooth; the similar statement for prismatic cohomology boils down to this fact. This follows from the fact that $\mathrm{L}_{\mathrm{B/A}} = 0$ for étale $\mathrm{A}$-algebras $\mathrm{B}$, so using the transitivity triangle étale locally we are looking at the cotangent complex of polynomial algebras. More precisely, factor a smooth map as

$$\mathrm{A} \to \mathrm{A}[\mathrm{X}_1, \ldots, \mathrm{X}_n] \to \mathrm{B}$$

where the last map is étale. The transitivity triangle then yields an exact triangle

$$\mathrm{B} \otimes \mathrm{L}_{\mathrm{A}[\mathrm{X}_1,\ldots,\mathrm{X}_n]/\mathrm{A}} \to \mathrm{L}_{\mathrm{B/A}} \to 0.$$

Thus, the first map is an isomorphism and we appeal to $\Omega^1_{\mathrm{B/A}} \simeq \mathrm{B} \otimes \Omega^1_{\mathrm{A}[\mathrm{X}_1,\ldots,\mathrm{X}_n]/\mathrm{A}}$ by the similar exact sequence for Kähler differentials.

We are now ready for the definition of derived prismatic cohomology.

Definition 2.7. Let $(\mathrm{A}, \mathrm{I})$ be a bounded prism. The *derived prismatic cohomology* functor

$$\mathrm{R} \mapsto \mathrm{L}{\Delta}_{\mathrm{R/A}}$$

on derived $p$-complete simplicial $\mathrm{A/I}$-algebras $\mathrm{R}$ is defined as the non-abelian derived functor of ${\Delta}_{\mathrm{R/A}}$ on $\mathsf{Poly}^\wedge_{\mathrm{A/I}}$, with $\mathcal{C}$ as the category of objects in $\mathrm{D}_{(p,\mathrm{I})-\mathrm{comp}}(\mathrm{A})$ equipped with a $\phi_\mathrm{A}$-semilinear endomorphism.

Specializing to derived $p$-complete classical rings, this admits a natural globalization ${\Delta}_{\mathfrak{X}/\mathrm{A}}$ to $p$-adic formal schemes.

Remark 2.8. It is important that we use $\mathrm{D}_{(p,\mathrm{I})-\mathrm{comp}}(\mathrm{A})$ rather than $\mathrm{D(A)}$. Although on polynomial algebras the functor lands in this subcategory and the subcategory has all colimits, these colimits *do not* commute with the inclusion to $\mathrm{D(A)}$.

> In Bhatt's notes, he uses $\mathsf{Poly}_{A/I}$, but this leads to some additional technical details later when it is not longer obvious $L\Delta_{R/A} = \Delta_{R/A}$ on a $p$-completed polynomial ring. I've swapped this issue for hiding technical details in what happens for animation on $p$-complete polynomial algebras.

Immediately after making this extension, we can obtain a derived Hodge-Tate comparison that will fix the defect in the previous example.

> **Lemma 2.9 (Derived Hodge-Tate).** The complex $L\overline{\Delta}_{R/A}$ admits an exhaustive $\mathbf{N}$-indexed increasing filtration $\mathrm{Fil}_*^{\mathrm{conj}}$ with
> $$\mathrm{gr}_i^{\mathrm{conj}}(L\overline{\Delta}_{R/A}) \simeq (\Lambda^i L_{R/\bar{A}}\{-i\}[-i])_p^\wedge$$
> where the $p$-completion is derived.

*Proof.* Consider the case where $R \in \mathsf{Poly}_{A/I}^\wedge$. We can put a truncation filtration on $\overline{\Delta}_{R/A}$, and via Hodge-Tate it has graded pieces[1]

$$\Omega^i_{R/(A/I)}\{-i\}[-i].$$

In particular, we can upgrade $\overline{\Delta}_{R^\wedge/A}$ to an object in the filtered derived category

$$DF(A/I) := \mathsf{Fun}(\mathbf{Z}^{\mathrm{op}}, D_{\mathrm{comp}}(A/I)).$$

We will still want to additionally enforce derived completeness to carry out the Kan extension, for the same reasons as the previous remark.

Deriving this filtered enhancement, we obtain for general $R$ a lift of $L\overline{\Delta}_{R/A}$ in $DF(A/I)$ (by this I mean you can forget the filtration and get the original object). Note that actually everything still gets an $R$-module structure; this is because it holds on polynomial rings, and we get a $P^\bullet$-module structure upon taking a resolution. This becomes an $R$-module structure.

Passing to underlying objects in $D(A/I)$ or taking the associated graded $\mathrm{gr}_*$ in the filtered derived category preserves all colimits. Thus, the associated graded will have pieces

$$\mathrm{gr}_i^{\mathrm{conj}} \simeq L(\Omega^i_{-/(\bar{A})}\{-i\}[-i]),$$

which almost what we want. One has to check that when we do the non-abelian derived functor procedure on $\mathsf{Poly}_{A/I}^\wedge$ instead of $\mathsf{Poly}_{A/I}$, we get the derived $p$-completion of the cotangent complex. After this, the claim follows by noting that in this case (as well as the usual one without $p$-completions) derived functors of $\Omega^i_{-/\bar{A}}$ are $L\Lambda^i L_{-/\bar{A}}$. One can check this by writing down polynomial resolutions to compute both, since

[1] As Kush pointed out this is why we call it the conjugate filtration: if we do the same thing in de Rham cohomology, it is opposite of the Hodge filtration.

on these we get projective modules after passing through $\mathrm{L}_{-/\bar{\mathrm{A}}}$ (we'll get polynomial algebras) and then $\mathrm{L}\Lambda^i$ will just be the usual exterior powers. So a polynomial resolution computes both of these functors as the same sifted colimit over $\Delta^{\mathrm{op}}$. □

**Proposition 2.10 (Consequences of the derived Hodge-Tate comparison).** The following are also true for derived prismatic cohomology:

- On $p$-completely smooth algebras, the value of prismatic cohomology is unchanged.
- The functor $\mathrm{R} \mapsto \mathrm{L}{\Delta}_{\mathrm{R}/\mathrm{A}}$ is a sheaf for the $p$-completely étale topology on the category of derived $p$-complete $\mathrm{A}/\mathrm{I}$-algebras.
- The functor $\mathrm{R} \mapsto \mathrm{L}{\Delta}_{\mathrm{R}/\mathrm{A}}$ is also a sheaf for the quasisyntomic topology.
- The formation of $\mathrm{L}{\Delta}_{\mathrm{R}/\mathrm{A}}$ commutes with base change.
- For a perfect prism, we have a comparison to derived crystalline cohomology.

*Proof.* The idea for all of these is to use that $\mathrm{L}{\Delta}_{\mathrm{R}/\mathrm{A}}$ is derived $(p, \mathrm{I})$-complete, and $- \otimes^{\mathrm{L}} \mathrm{A}/\mathrm{I}$ is conservative (reflects isomorphisms). That is, we can test isomorphisms after passing to $\mathrm{L}\overline{\Delta}_{\mathrm{R}/\mathrm{A}}$ (this is essentially derived Nakayama).

We will show (1) in detail. To see that the value is unchanged on smooth algebras, it then suffices to check this for $\mathrm{L}\overline{\Delta}_{\mathrm{R}/\mathrm{A}}$ when $\mathrm{R}$ is $p$-completely smooth. By the universal property of Kan extensions, we have a map from

$$\mathrm{L}\overline{\Delta}_{\mathrm{R}/\mathrm{A}} \to \overline{\Delta}_{\mathrm{R}/\mathrm{A}}$$

compatibly with filtrations by using that this lifts to a morphism in $\mathrm{DF}(\mathrm{A}/\mathrm{I})$.

The associated graded functor $\mathrm{gr}_* : \mathrm{DF}(\mathrm{A}/\mathrm{I}) \to \mathrm{D}(\mathrm{A}/\mathrm{I})$ is conservative. But we know the associated graded of both sides for a $p$-completely smooth algebra, and the map induced by the universal property will give the map

$$\mathrm{L}\Lambda^i \mathrm{L}_{\mathrm{R}/\bar{\mathrm{A}}}\{-i\}[-i]^{\wedge} \to \Omega^i_{\mathrm{R}/\bar{\mathrm{A}}}$$

again induced by the universal property of Kan extensions. However, we know for the cotangent complex this is an isomorphism, and similarly $\mathrm{L}\Lambda^i : \mathsf{Mod}_{\mathrm{A}/\mathrm{I}} \to \mathrm{D}(\mathrm{A}/\mathrm{I})$ does not change its values on flat modules (which will be the case here: $\Omega^1$ is a projective module).

For quasisyntomic descent, by the same strategy the descent isomorphism can be reduced to testing for the cotangent complex which has flat descent.

The other claims are similar or easy to prove using the same general strategy to reduce to properties of the cotangent complex. ☐

I will now drop the $\mathrm{L}$, as we know the values agree on smooth algebras.

In the previous counterexample to the comparison theorems, the value of prismatic cohomology will actually be unchanged. However, this Hodge-Tate comparison allows for the graded pieces to "pile up" in a single degree, allowing for infinite dimensional cohomology. It is in fact possible to explicitly describe the first component of this filtration.

> Proposition 2.11 (Describing the first piece). There is a canonical isomorphism
> $$\alpha_{\mathrm{X}} : \mathrm{Fil}_1^{\mathrm{conj}} \simeq \mathrm{L}_{\mathrm{X}/\mathrm{A}}\{-1\}[-1]^{\wedge}.$$

*Sketch.* This argument is taken from *Prismatic Dieudonné theory* by Anschütz and Le Bras.

First, we may assume $\mathrm{X} = \mathrm{Spf}(\mathrm{R})$ is affine by descent. Using the transitivity triangle for $\mathrm{A} \to \bar{\mathrm{A}} \to \mathrm{R}$, we obtain

$$\mathrm{R} \otimes_{\bar{\mathrm{A}}} \mathrm{L}_{\bar{\mathrm{A}}/\mathrm{A}}\{-1\}[-1]^{\wedge} \longrightarrow \mathrm{L}_{\mathrm{R}/\mathrm{A}}\{-1\}[-1]^{\wedge} \longrightarrow \mathrm{L}_{\mathrm{R}/\bar{\mathrm{A}}}\{-1\}[-1]^{\wedge}$$

or just

$$\mathrm{gr}_0^{\mathrm{conj}} \longrightarrow \mathrm{L}_{\mathrm{R}/\mathrm{A}}\{-1\}[-1]^{\wedge} \longrightarrow \mathrm{gr}_1^{\mathrm{conj}}.$$

It then already looks like the correct thing to consider (which hopefully removes some mystery from the statement).

To construct $\alpha_{\mathrm{X}}$, it suffices to prove the case when $\bar{\mathrm{A}} \to \mathrm{R}$ is $p$-completely smooth and Kan extend. We will therefore restrict ourselves to the smooth affine case in what follows.

Let us now define $\alpha_{\mathrm{X}}$. Using smoothness, essentially want to produce an isomorphism

$$\alpha_{\mathrm{R}} : \mathrm{L}^{\wedge}_{\mathrm{R}/\mathrm{A}} \to (\tau^{\leq 1}\overline{\mathbb{\Delta}}_{\mathrm{R}/\mathrm{A}})\{1\}[1]$$

which can be done by producing a map to $\mathrm{B}/\mathrm{J}\{1\}[1]$ for each prism in the prismatic site taking the colimit. Here, we're really using smoothness to ensure we can compute prismatic cohomology site-theoretically.

The first way of providing this map is very simple. Given an object $(R \to B/J \leftarrow B)$ of the prismatic site $(R/A)_{\Delta}$, we have a natural map

$$L^{\wedge}_{R/A} \to L^{\wedge}_{(B/J)/B}$$

via the square

$$\begin{array}{ccc} R & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ A & \longrightarrow & B. \end{array}$$

Note $B/J$ cuts out an effective Cartier divisor or a regular codimension one embedding. In particular, it is lci. This yields

$$L_{(B/J)/B} \simeq J/J^2[1] \simeq B/J\{1\}[1].$$

Thus, we get the map! We can drop the completion since $B$ defines a prism. Note that if $R = A/I$, we get the map from the base extension $I/I^2 \to J/J^2$.

There is also a way to write down this map in terms of Ext groups of the cotangent complex parameterizing deformation theory problems. For $(B, J) \in (R/A)_{\Delta}$, we have an extension

$$0 \longrightarrow J/J^2 \longrightarrow B/J^2 \longrightarrow B/J \longrightarrow 0.$$

By definition of the prismatic site, there is a map $\iota : R \to B/J$. Thus, we obtain maps of extension groups

$$\mathrm{Ext}^1(B/J, J/J^2) \to \mathrm{Ext}^1(R, J/J^2) \to \mathrm{Ext}^1(A, J/J^2).$$

Using these extensions $E$ and $E'$ depending coming from our original extension $B/J^2$, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J/J^2 & \longrightarrow & E & \longrightarrow & R & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & J/J^2 & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Such a diagram is precisely the solution of a deformation problem, and is classified by an element of

$$\mathrm{Ext}^1(L_{R/A}, J/J^2) \simeq \mathrm{Ext}^1(L_{R/A}, B/J\{1\}).$$

That is, we again get a natural morphism

$$L^{\wedge}_{R/A} \to B/J\{1\}[1].$$

Taking a homotopy limit defines a map

$$\alpha_{\mathrm{R}} : \mathrm{L}^{\wedge}_{\mathrm{R/A}} \to \tau^{\leq 0}(\overline{\Delta}_{\mathrm{R/A}}\{1\}[1]) \simeq (\tau^{\leq 1}\overline{\Delta}_{\mathrm{R/A}})\{1\}[1].$$

Really this is a fancy way of saying we took derived global sections; the truncation is because the cotangent complex lies in $\mathrm{D}^{\leq 0}$. The cohomology sheaves of both sides are given by $\mathrm{R}\{1\}$ in degree $-1$ and $\Omega^1_{\mathrm{R/A}}$ in degree $0$. It suffices to show that $\alpha_{\mathrm{R}}$ induces the same maps as the Hodge-Tate comparison would on cohomology sheaves to get a functorial quasi-isomorphism in the smooth case.

We reduce to the case of $\mathbf{A}^1 = \mathrm{Spf}(\overline{\mathrm{A}}\langle x\rangle)$. This is enough, as it will also similarly prove the case of $\mathbf{A}^n$ (the only difference is you need to add more generators for $\Omega^1$) and we can étale localize to this due to working in the smooth case. Base change can reduce to $\mathrm{A} = \mathbf{Z}_p$, but this isn't necessary.

In this case, one explicitly has

$$\mathrm{L}^{\wedge}_{\mathrm{R/A}} \simeq \mathrm{R} \otimes_{\bar{\mathrm{A}}} \mathrm{I}/\mathrm{I}^2[1] \oplus \mathrm{R}\mathrm{d}x.$$

On the first summand, the map $\alpha_{\mathrm{R}}$ is easy to describe for a prism $(\mathrm{B}, \mathrm{J})$: it is given by the base extension $\mathrm{I}/\mathrm{I}^2 \to \mathrm{J}/\mathrm{J}^2$. On $\mathrm{R}\mathrm{d}x$, the morphism factors as

$$\mathrm{R} \xrightarrow{\ \iota\ } \mathrm{B}/\mathrm{J} \longrightarrow \mathrm{B}/\mathrm{J}\{1\}[1]$$

where the second morphism is the connecting morphism for $0 \to \mathrm{B}/\mathrm{J}\{1\} \to \mathrm{B}/\mathrm{J}^2 \to \mathrm{B}/\mathrm{J} \to 0$.

Now we pass to the limit with this description, and look at what happens for $\mathrm{H}^0$ (so we will study the $\mathrm{R}\mathrm{d}x$ component). We obtain a diagram

$$\begin{array}{ccc} \mathrm{R} & & \\ \big\downarrow{\scriptstyle \iota} & \searrow & \\ \overline{\Delta}_{\mathrm{R/A}} & \longrightarrow & \overline{\Delta}_{\mathrm{R/A}}\{1\}[1] \end{array}$$

On $\mathrm{H}^0$, by construction of the Bockstein the horizontal morphism induces

$$\beta_{\mathrm{I}} : \mathrm{H}^0(\overline{\Delta}_{\mathrm{R/A}}) \to \mathrm{H}^1(\overline{\Delta}_{\mathrm{R/A}})\{1\}.$$

Thus, on $\mathrm{H}^0$ we map $\mathrm{d}x \mapsto \beta_{\mathrm{I}}(\iota(x))$. In particular, $\alpha_{\mathrm{R}}$ will induce the identity after making identifications $\mathrm{H}^0(\mathrm{L}^{\wedge}_{\mathrm{R/A}}) \simeq \Omega^{1,\wedge}_{\mathrm{R/A}}$ and $\mathrm{H}^0(\overline{\Delta}_{\mathrm{R/A}}\{1\}[1]) \simeq \Omega^{1,\wedge}_{\mathrm{R/A}}$ (this is Hodge-Tate: the identification is $f\mathrm{d}x \mapsto f\beta_{\mathrm{I}}(x)$).

Next, we check $\mathrm{H}^{-1}$. This is easier, as $\alpha_{\mathrm{R}}$ is simply base extension and we see the map

$$\mathrm{H}^{-1}(\mathrm{L}^{\wedge}_{\mathrm{R/A}}) \simeq \mathrm{R} \otimes_{\bar{\mathrm{A}}} \mathrm{I}/\mathrm{I}^2 \to \mathrm{H}^{-1}(\overline{\Delta}_{\mathrm{R/A}}\{1\}[1])$$

is the canonical one given by twisting the structure map $\mathrm{R} \to \mathrm{H}^0(\overline{\Delta}_{\mathrm{R}/\mathrm{A}})$ (so in particular it also matches Hodge-Tate).

Thus, on cohomology sheaves we see that $\alpha_{\mathrm{R}}$ yields the same maps as the Hodge-Tate comparison. We can then conclude that $\alpha_{\mathrm{R}}$ induces the canonical Hodge-Tate comparison isomorphisms on cohomology sheaves for general smooth algebras, which concludes the proof. ☐

**Theorem 2.12 (Discrete examples).** Assume that $(\mathrm{A}, \mathrm{I})$ is a bounded prism, and assume that $\mathrm{R}$ is a derived $p$-complete simplicial $\mathrm{A}/\mathrm{I}$-algebra such that $\overline{\Delta}_{\mathrm{R}/\mathrm{A}}$ is concentrated in degree zero.

The following hold:

(1) The $\varphi_{\mathrm{A}}$-linear Frobenius $\varphi_{\mathrm{R}}$ on $\Delta_{\mathrm{R}/\mathrm{A}}$ naturally lifts to a $\delta$-$\mathrm{A}$ structure on the ring $\Delta_{\mathrm{R}/\mathrm{A}}$.

(2) The pair $(\Delta_{\mathrm{R}/\mathrm{A}}, \mathrm{I}\Delta_{\mathrm{R}/\mathrm{A}})$ gives a prism over $(\mathrm{A}, \mathrm{I})$ equipped with a map

$$\mathrm{R} \to \overline{\Delta}_{\mathrm{R}/\mathrm{A}} = \Delta_{\mathrm{R}/\mathrm{A}}/\mathrm{I}\Delta_{\mathrm{R}/\mathrm{A}}.$$

(3) The category "$(\mathrm{R}/\mathrm{A})_{\Delta}$" of prisms $(\mathrm{B}, \mathrm{J})$ over $(\mathrm{A}, \mathrm{I})$ equipped with a map

$$\mathrm{R} \to \mathrm{B}/\mathrm{J}$$

has an initial object. Moreover, the initial object is the image of an idempotent endomorphism of $(\Delta_{\mathrm{R}/\mathrm{A}}, \mathrm{I}\Delta_{\mathrm{R}/\mathrm{A}})$.

*Proof.* First, note that $\Delta_{\mathrm{R}/\mathrm{A}}$ is also concentrated in degree zero by applying derived Nakayama to the map $\mathrm{H}^0(\Delta_{\mathrm{R}/\mathrm{A}}) \to \Delta_{\mathrm{R}/\mathrm{A}}$.

For item (1), it will be easiest to interpret a $\delta$ structure on a simplicial ring $\mathcal{R}$ as map $w : \mathcal{R} \to \mathrm{W}_2(\mathcal{R})$ so $\varepsilon \circ w = \mathrm{id}$ ($\varepsilon$ drops the second coordinate of $\mathrm{W}_2$, so $w(x) = (x, \delta(x))$).

To obtain a simplicial version of the Čech-Alexander complex, first consider a polynomial resolution $\mathrm{P}^\bullet$ of the simplicial $\mathrm{A}/\mathrm{I}$-algebra $\mathrm{R}$. This can be done functorially. We then have

$$\Delta_{\mathrm{R}/\mathrm{A}} \simeq |\Delta_{\mathrm{P}^\bullet/\mathrm{A}}| = \mathrm{colim}_{\Delta^{\mathrm{op}}} \Delta_{\mathrm{P}^\bullet/\mathrm{A}}.$$

where the polynomial $\mathrm{A}/\mathrm{I}$-algebras in $\mathrm{P}^\bullet$, if chosen functorially, usually have infinitely many generators. Prismatic cohomology on these is a colimit of cohomologies of finitely generated polynomial algebras.

We have a Čech-Alexander complex computing $\Delta_{\mathrm{P}^\bullet/\mathrm{A}}$, which again can be done functorially. These are *cosimplicial* algebras, and to obtain the actual complex one takes totalization: this is a limit over $\Delta$.

It follows that

$$\Delta_{R/A} \simeq \operatorname{colim}_{\Delta^{op}} \lim_{\Delta} F_A(R)$$

where $F_A(R)$ is a simplicial object that assembles the Čech-Alexander complexes for $P^\bullet$. That is, $F_A(R)$ is a *simplicial cosimplicial* derived $(p, I)$-complete $\delta$-A-algebra. The cosimplicial comes from the Čech-Alexander complexes for individual $P^i$ which are cosimplicial $\delta$-A-algebras, but these $P^i$ come from a simplicial polynomial algebra $P^\bullet$ and so we assemble them into a simplicial cosimplicial object.

The point of all of this is that $F_A(R)$ has a $\delta$-A-algebra structure classified by a map

$$F_A(R) \to W_2(F_A(R)).$$

On the right, this means you apply $W_2$ termwise as everything is termwise a $\delta$-A-algebra.

This morphism lies over $A \to W_2(A)$ and splits $W_2(F_A(R)) \to F_A(R)$ (this is the $\varepsilon$ part of the characterization). Now just applying colimits and limits to everything, we get a map

$$\Delta_{R/A} \to \operatorname{colim}_{\Delta^{op}} \lim_{\Delta} W_2(F_A(R))$$

lying over $A \to W_2(A)$, and splitting

$$\operatorname{colim}_{\Delta^{op}} \lim_{\Delta} W_2(F_A(R)) \to \Delta_{R/A}.$$

**Claim**. We have $\operatorname{colim}_{\Delta^{op}} \lim_{\Delta} W_2(F_A(R)) \simeq W_2(\Delta_{R/A})$.

To see this, note that there is a *functorial* pullback square

$$\begin{array}{ccc} W_2(B) & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi} & B \otimes^{L}_{\mathbf{Z}} \mathbf{F}_p \end{array}$$

which also holds in simplicial rings. In particular, we get a pullback square

$$\begin{array}{ccc} \lim_{\Delta} W_2(F_A(R)) & \xrightarrow{F} & \lim_{\Delta} F_A(R) \\ \downarrow & & \downarrow \\ \lim_{\Delta} F_A(R) & \xrightarrow{\varphi} & (\lim_{\Delta} F_A(R)) \otimes^{L}_{\mathbf{Z}} \mathbf{F}_p \end{array}$$

We used here that the tensor product with $\mathbf{F}_p$ commutes with limits. We were able to pull in $W_2$ as it's defined termwise.

Next, we apply a colimit over $\Delta^{op}$. To argue this produces a cartesian square, we will need to appeal to some facts about $\infty$-categories.

The cartesian square can be written as a finite limit. Thus, the claim that $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ preserves cartesian squares in an $\infty$-category $\mathcal{C}$ would follow from a more general claim that sifted colimits (e.g. colimits indexed by $\Delta^{\mathrm{op}}$) commute with pullback squares.

**Fact**. If $\mathcal{C}$ is a stable $\infty$-category this is true (basically by definition) since pullback squares are pushout squares.

Unfortunately, in our case the category of simplicial commutative rings is not stable. However, the category of (animated) simplicial commutative $\mathrm{A}$-algebras is Grothendieck prestable by SAG C.1.5.7 applied to $\mathsf{Poly}_{\mathrm{A}}$. In particular, every pushout square is a pullback square by SAG C.1.2.6. Thus sifted colimits will preserve pullback squares.

We therefore get a pullback square again after applying $\mathrm{colim}_{\Delta^{\mathrm{op}}}$. Our new square is

$$\begin{array}{ccc} \mathrm{colim}_{\Delta^{\mathrm{op}}} \lim_{\Delta} \mathrm{W}_2(F_{\mathrm{A}}(\mathrm{R})) & \xrightarrow{F} & \mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \\ \downarrow & & \downarrow \\ \mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} & \xrightarrow{\varphi} & \mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathbf{Z}} \mathbf{F}_p. \end{array}$$

Here, we used the same argument on the bottom right. Revisiting the maps we wrote earlier with this identification, by discreteness we have precisely produced a $\delta$-$\mathrm{A}$-algebra structure on $\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}$, proving (1).

Part (2) is then immediate from derived $(p, \mathrm{I})$-completeness and discreteness of $\overline{\mathbb{\Delta}}_{\mathrm{R}/\mathrm{A}}$. To deduce the claim, we need $\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}$ to be a discrete derived $(p, \mathrm{I})$-complete $\delta$-$\mathrm{A}$-algebra without $\mathrm{I}$-torsion to apply Lemma 3.5 in the paper (rigidity of prisms). Observe $\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}} \mathrm{A}/\mathrm{I}^n$ must also be discrete due reducing to $\mathrm{I} = (d)$ and applying the distinguished triangle

$$\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{I}^n \longrightarrow \mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{I}^{n+1} \longrightarrow \overline{\mathbb{\Delta}}_{\mathrm{R}/\mathrm{A}}.$$

Since the Mittag-Leffler condition is satisfied the $\mathrm{R}\lim$ will be discrete. We don't get any I-torsion since reducing to $\mathrm{I} = (d)$, we see $\mathrm{H}^1(\overline{\mathbb{\Delta}}_{\mathrm{R}/\mathrm{A}})$ is $\mathrm{Tor}_1^{\mathrm{A}}(\mathrm{A}/d, \mathbb{\Delta}_{\mathrm{R}/\mathrm{A}})$ which is isomorphic to the $d$-torsion.

For part (3) we need to do slightly more work. We will first construct a functorial map

$$(\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}, \mathrm{I}\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}) \to (\mathrm{B}, \mathrm{J})$$

for any $(\mathrm{B}, \mathrm{J}) \in (\mathrm{R}/\mathrm{A})_{\mathbb{\Delta}}$ (we use the same meaning as in the statement of (3); recall $\mathrm{R}$ need not be a discrete ring here).

Fix a resolution $\mathrm{P}^\bullet \to \mathrm{R}$ of $p$-completely ind-smooth $\mathrm{A}/\mathrm{I}$-algebras (we can even take these to be ind-polynomial). Now we have

$$\Delta_{\mathrm{R}/\mathrm{A}} = \mathrm{colim}_{\Delta^{\mathrm{op}}} \lim_{C \in (\mathrm{P}^\bullet/\mathrm{A})_\Delta} C$$

in the category of derived $(p, \mathrm{I})$-complete complexes. This is because $\lim_{C \in (\mathrm{P}^i/\mathrm{A})_\Delta} C$ computes $\Delta_{\mathrm{P}^i/\mathrm{A}}$ in the smooth affine case, as we may use the indiscrete topology (really Čech-Alexander theory tells us we get the same answer).

Now $(\mathrm{B}, \mathrm{J})$ (like any other prism we are considering in $(\mathrm{R}/\mathrm{A})_\Delta$) yields compatible objects in $(\mathrm{P}^\bullet/\mathrm{A})_\Delta$, which by abuse of notation we all denote by $\mathrm{B}$. This induces a map

$$\Delta_{\mathrm{R}/\mathrm{A}} \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} \lim_{C \in (\mathrm{P}^\bullet/\mathrm{A})_\Delta} C \to \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{B} = \mathrm{B}.$$

Repeating the argument for (1) shows we get a map of $\delta$-rings. Thus, we have shown that $(\Delta_{\mathrm{R}/\mathrm{A}}, \mathrm{I}\Delta_{\mathrm{R}/\mathrm{A}})$ is weakly initial.

To see we get an initial object via an idempotent endomorphism, we appeal to the following categorical lemma.

> Lemma 2.13. Let $\mathcal{C}$ be an idempotent complete category and $\mathrm{X} \in \mathcal{C}$ an object. Let $\mathcal{C}_{\mathrm{X}\backslash}$ be the category of objects $\mathrm{Y}$ with a morphism $\mathrm{X} \to \mathrm{Y}$.
>
> Assume $\mathrm{id}_{\mathcal{C}}$ factors over
> $$\mathcal{C}_{\mathrm{X}\backslash} \to \mathcal{C}$$
> via $F : \mathcal{C} \to \mathcal{C}_{\mathrm{X}\backslash}$. Then $F(\mathrm{X})$ yields an idempotent endomorphism of $\mathrm{X}$, and the corresponding retract (image) is an initial object of $\mathcal{C}$.

Apply this to $\mathcal{C} = (\mathrm{R}/\mathrm{A})_\Delta$ and $\mathrm{X} = (\Delta_{\mathrm{R}/\mathrm{A}}, \mathrm{I}\Delta_{\mathrm{R}/\mathrm{A}})$. The factoring of the identity comes from being weakly initial. □

Finally, we will end with an example of computing prismatic cohomology.

Let $(\mathrm{A}, \mathrm{I})$ be a bounded prism, and set

$$\mathrm{R} := \mathrm{A}/(\mathrm{I}, f_1, \dots, f_r)$$

where $f_i$ are Koszul-regular on $\mathrm{A}/\mathrm{I}$. Assume this has bounded $p$-torsion.

**Claim**. $\Delta_{\mathrm{R}/\mathrm{A}}$ is concentrated in degree $0$ and $\mathrm{I}$ torsionfree.

This follows from the Hodge-Tate comparison. We know we have a conjugate filtration of $\overline{\Delta}_{\mathrm{R}/\mathrm{A}}$ given by $\Lambda^i \mathrm{L}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})}\{-i\}[-i]^\wedge$. As $\mathrm{R}$ is given by a quotient of $\mathrm{A}/\mathrm{I}$ by a Koszul-regular sequence $f_i$ (generating $\mathrm{J} = (f_i)$), we have

$$\mathrm{L}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})} \simeq (\mathrm{J}/\mathrm{J}^2)[1].$$

Looking at the derived exterior powers, we'll get $\Gamma^i_{A/I}(J/J^2)[i] = ((J/J^2)^{\otimes i})^{\Sigma_i}$. Thus all of these components of the conjugate filtration will be concentrated in degree zero, and then claim then follows.

Now we are in the situation of the previous theorem, and so $(\Delta_{R/A}, I\Delta_{R/A})$ is a weakly initial object in $(R/A)_{\Delta}$.

> PROPOSITION 2.14. Let $R = A/(I, f_1, \dots, f_r)$ where $f_i$ are Koszul-regular on $A/I$ as before. The prism $(\Delta_{R/A}, I\Delta_{R/A})$ is initial in $(R/A)_{\Delta}$, and $\Delta_{R/A} \simeq A\left\{\frac{f_i}{d}\right\}^{\wedge}_{(p,d)}$.

> REMARK 2.15. When the prism is perfect, this example is particularly important: this shows that the prismatic cohomology of a quasiregular semiperfectoid ring is $\mathbf{A}_{\mathrm{inf}}(R)\{\frac{f_i}{d}\}^{\wedge}$.

*Proof.* This is a local assertion on $\operatorname{Spf} A$, so by ind-Zariski localization we may assume that $I = (d)$. Define

$$B := A\left\{\frac{f_i}{d}\right\}^{\wedge}_{(p,d)}$$

We have a natural map $B \to \Delta_{R/A}$; specifying a map from $B$ into a $d$-torsionfree $\delta$-$A$-algebra $\Delta_{R/A}$ amounts to specifying a map $A \to \Delta_{R/A}$ (the structure map) so that it carries the $f_i$ land in $d\Delta_{R/A}$. In $\overline{\Delta}_{R/A}$, by the previous calculation the $f_i$ vanish. Thus we indeed have a map.

If we can show that the natural map

$$B \to \Delta_{R/A}$$

is an isomorphism, we learn that $B$ is discrete so $(B, dB) \in (R/A)_{\Delta}$. We also learn that this prism $(B, dB)$ must be initial. Given a prism $(A', dA')$ in $(R/A)_{\Delta}$ (so it has a map $R \to A'/d$) to specify a map $(B, dB) \to (A', dA')$ amounts to a map $A \to A'$ sending the $x_i$ into $(d)$. The existence of $R = A/(d, x_1, \dots, x_r) \to A'/d$ forces the condition on the $x_i$ to hold, so there is a unique map $A \to A'$ induced by this. Thus, such an isomorphism would tell us that the initial object of $(R/A)_{\Delta}$ is $(B, dB) \simeq (\Delta_{R/A}, d\Delta_{R/A})$, which is everything we wanted.

Now we turn to showing $B \to \Delta_{R/A}$ is an isomorphism. It suffices to check in the universal case by base change. Namely, you can put $A = \mathbf{Z}_p\{d, f_1, \dots, f_r\}[\delta(d)^{-1}]^{\wedge}$. Here $A/d$ and $R$ are $p$-torsionfree.

In this case $B = A\left\{\frac{f_i}{d}\right\}^{\wedge}_{(p,d)}$ identifies as the prismatic envelope of $(A, J = (d, f_1, \dots, f_r))$, after checking the $(p, d)$-complete regularity condition. It follows $B$ is discrete, and by the same reasoning it is initial.

By part (3) of the previous theorem, the map $B \to \Delta_{R/A}$ admits a retraction (i.e. we get $B \to \Delta_{R/A} \to B$ with composite equal to the identity). It follows $B/dB \to \overline{\Delta}_{R/A}$ also admits a retraction. The cokernel of this map will be $p$-torsionfree as $\overline{\Delta}_{R/A}$ is (since $R$ is). We may therefore just check that

$$(B/dB)[1/p] \twoheadrightarrow \overline{\Delta}_{R/A}[1/p],$$

as this will imply the cokernel is trivial and then the claim follows.

By Hodge-Tate, as it is discrete we know $\overline{\Delta}_{R/A}[1/p]$ is generated as a (Banach) $R[1/p]$-algebra by $\mathrm{gr}_1 = J/(J^2 + I)\{-1\}$. It is therefore sufficient to show that

$$B = A\left\{\frac{J}{I}\right\} \to \Delta_{R/A}$$

maps $\frac{J}{I}$ surjectively onto $\mathrm{gr}_1$. This is not too hard to check, since you basically need to see that $\frac{f_i}{d}$ don't get sent to zero. $\square$